Generalized Nyquist Consensus Condition for Linear Multi Agent Systems with Heterogeneous Delays

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Abstract: We study robust consensus of linear Multi-Agent Systems (MAS) with heterogeneous delays. We provide a very accurate set-valued condition that guarantees consensus for arbitrary bounded delays and arbitrary connected topologies. This condition is proven using the generalized Nyquist criterion. From the set-valued condition, an explicit analytical formula is derived for single integrator MAS. In this publication, the delays affect both the agent’s own state and its neighbor’s state; and we assume that the delay of the agent’s own state and its neighbor’s state are different, e.g. due to a combination of computation delays, which affect both states, and transmission delays, which only affect the neighbor’s state.

Keywords: Multi-agent systems, delays, generalized Nyquist criterion.

1. INTRODUCTION

Multi-Agent Systems (MAS) are large groups of dynamical systems, called agents, that interact in order to achieve a cooperative behavior like consensus, rendezvous, or flocking. There is a broad range of applications for MAS, such as swarms of animals, distributed computing, or autonomous vehicle coordination, see Olfati-Saber et al. (2007) for an overview. This broad range of applications has attracted the attention of the control community.

A MAS model consists of two parts: a model for the agents, e.g. differential equations, and a model for the network between the agents. This network is usually described by a graph representing the topology of the network, and it is assumed that information is transmitted instantaneously between the agents. In technical MAS, the agents are often interconnected using a digital communication network. These networks introduce packet delays, packet loss and quantization. Therefore, it is important to investigate the influence of these network properties on the behavior of the MAS.

This paper presents a method to investigate the influence of delays on MAS behavior. More precisely, we derive a set-valued consensus condition for high-order linear MAS with arbitrary but bounded heterogeneous delays using the generalized Nyquist criterion, e.g. Desoer and Wang (1980). The simplicity and accuracy of these conditions is very surprising because the analysis of time-delay systems with heterogeneous delays is in general NP-hard, cf. Toker and Ozbay (1996). From these set-valued conditions, we derive explicit analytical formulas for single integrator MAS with communication delays. We study MAS where both the agent’s own state and the neighbor’s state are affected by different delays. This extends our previous work in Münz et al. (2009b), which applied the generalized Nyquist criterion to MAS where the agent’s one state is either not delayed or affected by the same delay as the neighboring agent. The generalized Nyquist criterion has also been used recently for network stability and consensus analysis in Lestas and Vinnicombe (2007a,b, 2006). Compared to Lestas and Vinnicombe (2007a), we use simple convex sets instead of s-hulls. Therefore, our conditions are usually easier to check at the expense of considering homogeneous instead of heterogeneous agent dynamics.

MAS without delays have been studied extensively in the past, see Jadabaie et al. (2003); Fax and Murray (2004); Olfati-Saber et al. (2007); Arcak (2007); Ren and Beard (2008); Wieland et al. (2008) and references therein. Yet, there are much less publications for MAS with delays. Single integrator MAS with heterogeneous delays have been analyzed in Papachristodoulou and Jadbabaie (2006); Ghabcheloo et al. (2007); Bliman and Ferrari-Trecate (2008); Münz et al. (2009a). Higher order agent dynamics are considered in Lee and Spong (2006); Chopra and Spong (2006, 2008); Yang et al. (2008); Münz et al. (2008, 2009c). Most publications on MAS with heterogeneous delays provide sufficient conditions for consensus to be robust to unbounded delays, e.g. Papachristodoulou and Jadbabaie (2006); Lee and Spong (2006); Chopra and Spong (2006, 2008); Ghabcheloo et al. (2007); Münz et al. (2009a,c). Yet, these delay-independent conditions cannot guarantee consensus if consensus is delay-dependent, i.e. consensus is only achieved for sufficiently small delays, see Münz et al. (2008) for an example. Therefore, we need accurate delay-dependent consensus conditions. This paper provides a delay-dependent condition for MAS where delays affect both the agent’s own state and its neighbor’s state and these delays are different. This delay configuration includes the single integrator MAS with computation and communication delays considered in Xiao and Wang (2008); Tian and Liu (2008); Liu and Tian (2007) as special case.

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The paper is structured as follows: The problem statement and preliminary results on the generalized Nyquist criterion are summarized in Section 2. The set-valued consensus condition for linear MAS with delays is presented in Section 3. The exact delay bound for first order MAS is derived in Section 4 before the paper is concluded in Section 5. Technical proofs are given in Section 6.

2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

2.1 Problem Statement

We analyze the consensus properties of linear MAS with delayed communication. We consider the following agent dynamics

\[ x_i^{(p)}(t) = - \sum_{j=0}^{p-1} \alpha_i N \sum_{k=1}^{N} \frac{a_{ji}}{d_i} x_j^{(k)}(t - T_{ji}) - x_j^{(k)}(t - \tau_{ji}). \]  

(1)

The agents are given as \( p \)-th order systems that exchange their complete state \( \xi_i(t) = [x_i^{(p-1)}(t), \ldots, x_i(t), x_i(t)]^T \in \mathbb{R}^p \) over a network. We assume that both delays are symmetric and bounded, i.e. \( T_{ji} = T_{ij} \leq \tau \) and \( \tau_{ji} = \tau_{ij} \leq \tau \). Motivating applications for non-identical self-delays \( T_{ji} \) and neighbor-delays \( \tau_{ji} \) include different reaction delays to the agent’s own behavior and the behavior of its neighbors or computation delays \( T_{ji} \) in combination with transmission delays, cf. Xiao and Wang (2008); Tian and Liu (2008); Liu and Tian (2007).

Each agent (1) seeks to reach a consensus with its neighbors by comparing its own state \( \xi_i \) to the weighted average of the states of its neighbors \( \xi_j \), which is modeled by the elements \( a_{ji} \geq 0 \) of the adjacency matrix \( A \) of the undirected communication graph and the degree \( d_i = \sum_{j=1}^{N} a_{ji} \) of agent \( i \). The influence of the different states on \( x_i^{(p)}(t) \) is determined by the coefficients \( \alpha_i \), which describe the dynamics of the MAS. For these coefficients, we have the following assumption:

**Assumption 1.** The polynomial \( s^p + \sum_{k=0}^{p-1} \alpha_i s^k \) is Hurwitz.

Assumption 1 is necessary for consensus in MAS like (1) on arbitrary connected graphs without delays, which can be shown based on the results in Wieland et al. (2008). Hence, it is also necessary for MAS on arbitrary graphs with arbitrary bounded delays \( \tau_{ji}, T_{ji} \leq \tau \). Next, we define consensus for MAS (1):

**Definition 2.** Consensus is reached asymptotically if

\[ \lim_{t \to \infty} \|x_i(t) - x_j(t)\| = 0 \quad \text{for all } i, j. \]  

(2)

Note that consensus implies \( x_i^{(k)}(t) - x_j^{(k)}(t) = 0 \) for the \( k \)-th derivative of \( x_i \) and \( x_j \) for all \( k \).

In this contribution, we present a condition for MAS (1) to reach a consensus asymptotically. The condition is based on the generalized Nyquist criterion, e.g. Desoer and Wang (1980), and provides simple set-valued conditions for consensus in arbitrary connected networks with arbitrary bounded delays \( \tau_{ji}, T_{ji} \leq \tau \). In Münn et al. (2009b), we studied MAS (1) without self-delay \( T_{ji} = 0, \forall i, j \) and with identical self-delay \( \tau_{ji} = \tau_{ij}, \forall i, j \). A comparison of the three different delay cases is given in Section 4.

2.2 Algebraic Graph Theory

The communication network between the agents is modeled by an undirected graph. A graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) consists of a set of vertices (nodes) \( \mathcal{V} = \{v_i, i \in \mathcal{N} = \{1, \ldots, N\} \) which represent the agents, and a set of edges (links) \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \), which represent the communication channels between the agents. If \( v_i, v_j \in \mathcal{V} \) and \( e_{ij} = (v_i, v_j) \in \mathcal{E} \), then there is an edge from node \( v_i \) to node \( v_j \), i.e. agent \( i \) and \( j \) can exchange data. In this paper, we assume that the graph \( \mathcal{G} \) is undirected, i.e. \( e_{ij} \in \mathcal{E} \) if and only if \( e_{ji} \in \mathcal{E} \). We also assume that the network topology does not contain self-loops, i.e. \( e_{ii} \notin \mathcal{E} \). The graph adjacency matrix \( A = [a_{ij}], A \in \mathbb{R}^{N \times N} \), is such that \( a_{ij} > 0 \) if \( e_{ij} \in \mathcal{E} \) and \( a_{ij} = 0 \) if \( e_{ij} \notin \mathcal{E} \). For the undirected graph, we have \( a_{ji} = a_{ij} \), i.e. \( A = A^T \). If \( e_{ij} \in \mathcal{E} \), then \( v_i \) and \( v_j \) are neighbors. The degree or valence of vertex \( v_i \) is denoted by \( d_i = \sum_{j=1}^{N} a_{ji} \). The diagonal valency matrix is \( D = \text{diag}(d_i) \) and the Laplacian matrix of the graph is \( L = D - A \). An undirected graph is connected if any two nodes of the graph are connected by a path, i.e. a sequence of neighboring nodes. More details on algebraic graph theory can be found for example in Godsil and Royle (2000).

2.3 Laplace Transform and Generalized Nyquist Criterion

We transform MAS (1) into the Laplace domain. We define the delay-dependent adjacency and valency matrices

\[ A(s) = [a_{ij} e^{-\tau_{ij}s}], \quad D_T(s) = \text{diag} \left( \sum_{j=1}^{N} a_{ij} e^{-T_{ij}s} \right), \]  

(3)

which combine the network’s topology and delay. With these matrices, we express the Laplace transform of (1) as

\[ s^p X(s) = - \sum_{k=0}^{p-1} \alpha_i s^k D^{-1}(D_T(s) - A(s))X(s), \]  

(4)

where \( X(s) = [x_1(s), \ldots, x_N(s)]^T \) and \( x_i(s) \) is the Laplace transform of \( x_i(t) \). The closed loop system achieves consensus if the characteristic equation of (1) satisfies

\[ \Delta(s) = \det \left( s^p + \sum_{k=0}^{p-1} \alpha_i s^k D^{-1}(D_T(s) - A(s)) \right) \neq 0 \]  

(5)

for all \( s \in \mathbb{C} : \Re(s) > 0, s \neq 0 \), i.e. there are no roots in the right half plane. Note that \( \Delta(s) \) has a root with multiplicity \( p \) at the origin if the graph is connected. The corresponding eigenvector is \( \mathbf{1} = [1, 1, \ldots, 1]^T \), which corresponds to the consensus subspace. The computation of the nonzero roots of (5) is extremely difficult because the matrices \( D_T \) and \( A(s) \) depend on \( s \). Instead of computing these roots, we may use the generalized Nyquist criterion as proposed, e.g., in Desoer and Wang (1980) to determine the stability of the closed loop. The main result can be summarized in simplified form:

**Theorem 3.** Consider a positive feedback system as illustrated in Figure 1 with real proper \( N \times N \) open-loop transfer function \( G(s) \). \( \Gamma = \{ \gamma_q, q = 1, \ldots, Q \} \) is the set of eigenloci of \( G(s) \) for \( s = \gamma_q \) when \( \omega \) moves from \( -\infty \) to \( +\infty \). Then, the closed-loop system is stable if and only if

\[ i) \quad +1 \notin \Gamma \quad \text{and} \]  

\[ ii) \quad \sum_{q=1}^{Q} \left( C(\gamma_q, v_1) = 0 \right. \]  

(6)

(7)

Subsection 2.2 gives a more detailed review on algebraic graph theory.
where $p^+_0$ is the number of right half plane poles of the open loop system $G$ and $C(+1, \gamma_q)$ denotes the number of clockwise encirclements of $+1$ by $\gamma_q$.

In order to apply Theorem 3 to the present consensus problem, we define the network return ratio of the MAS (1)

$$G(s) = \frac{\sum_{k=0}^{p-1} \alpha_k s^k}{s^p + \sum_{k=0}^{p-1} \alpha_k s^k} (I - D^{-1}(D_T(s) - A_\tau(s))), \quad (8)$$

which results directly from (4). Note that all poles of $G$ are in the left half plane because $s^p + \sum_{k=0}^{p-1} \alpha_k s^k$ is Hurwitz by Assumption 1.

Following the generalized Nyquist criterion, consensus is reached if the eigenloci of $G$ neither touch nor encircle the point $+1$ for $\omega \neq 0$. This is the main idea of this paper. The main challenges are the computation of appropriate sets that contain the spectrum of $G$. It is important to note that alternative approaches, e.g. using Gershgorin’s circle theorem as in Tian and Liu (2008), usually result in delay-independent consensus conditions. However, the method proposed here gives frequency- and delay-dependent consensus conditions for a much broader class of consensus problems.

3. CONSENSUS IN LINEAR MAS WITH DELAY

As discussed before, consensus is guaranteed if the eigenloci of $G(j\omega)$ neither touch nor encircle the point $+1$ to this end, we have to find a set that contains the eigenvalues of $I - D^{-1}(D_T(j\omega) - A_\tau(j\omega))$. The characterization of this set is the main contribution of this work. It is provided in the following lemma, which is proven in Section 6:

**Lemma 4.** The spectrum of $I - D^{-1}(D_T(j\omega) - A_\tau(j\omega))$ satisfies

$$\sigma(I - D^{-1}(D_T(j\omega) - A_\tau(j\omega))) \subset \Omega_1(\omega)$$

$$= \text{Co}\{1 - e^{-j\psi} + e^{-j\varphi}, 1 - e^{-j\psi} - e^{-j\varphi} : \psi, \varphi \in [0, \omega]\} \quad (9)$$

The set $\Omega_1(\omega)$ is illustrated in Figure 2 for some $\Omega_1$ and $\omega > 0$. The yellow part corresponds to $\text{Co}\{e^{-j\varphi}, e^{-j\psi} : \varphi \in [0, \omega]\}$ and the blue part to $\text{Co}\{1 - e^{-j\psi} : \psi \in [0, \omega]\}$. The set $\Omega_1$ results from a translation of the yellow set such that its center always remains inside the blue set. $\Omega_1(\omega)$ is the union of the green, yellow, and blue set. For sufficiently small $\omega$, $\Omega_1$ is a subset of a circle with radius 2 and center $+1$. The circle is complete for $\omega \pi \geq 2\pi$. Hence, the delay bound $\Omega$ determines how fast the circle is completed as $\omega$ increases.

In Figures 3 and 4, we present the corresponding sets $\Omega_2$ and $\Omega_3$ for MAS without self-delay, i.e. $T_{ji} = 0, \forall i, j$, and for MAS with identical delays $T_{ji} = \tau_{ji}, \forall i, j$, respectively.

![Figure 1. General positive feedback interconnection.](image1)

![Figure 2. Exemplary set $\Omega_1$ according to (9).](image2)

![Figure 3. Exemplary set $\Omega_2(T_{ji} = 0)$ from Münz et al. (2009b).](image3)

![Figure 4. Exemplary set $\Omega_3(T_{ji} = 0)$ from Münz et al. (2009b).](image4)

These special cases have been considered in Münz et al. (2009b). Note that the circle that contains $\Omega_2$ is centered at the origin and has radius 1, i.e. it is much smaller than the other circles.

Now, we state the main result of this paper:

**Theorem 5.** A MAS (1) with arbitrary size $N \in \mathbb{N}$, arbitrary delays $\tau_{ji}, T_{ji} \leq \Omega$, and arbitrary connected topology achieves...
consensus asymptotically if
\[
(j\omega)^p + \sum_{k=0}^{p-1} \alpha_k(j\omega)^k \notin \Omega_1(\omega)
\] (12)
for all \(\omega \in \mathbb{R} \setminus \{0\}\).

The proof is given in Section 6. The condition in Theorem 5 can be checked graphically for general \(p\) and \(\alpha_k\) or even analytically for particular \(p\) and \(\alpha_k\). The latter will be shown in the next section.

Remark 6. Condition (12) is only mildly conservative in the sense that if it does not hold for given \(\alpha_k, \tau\), then there exist in most cases a topology and delay values \(\tau_{ji}, \tau_{ji} \leq \tau\) such that consensus is not reached. This is due to the fact that, for arbitrary \(N\), topologies, and delays \(\tau_{ji}, \tau_{ji} \leq \tau\), the eigenvalues of \(I - D^{-1}(D_T(j\omega) - A_T(j\omega))\) can take almost any value in \(\Omega_3(\omega)\). This can be seen from numerical computations. Exemplary, we show the eigenvalues of \(I - D^{-1}(D_T(j\omega) - A_T(j\omega))\) with \(\omega = 3\) for a particular graph with 10 nodes in Figure 5.

The black dots indicate the eigenvalues of this matrix for 1000 randomly chosen delay sets \(\tau_{ji}, \tau_{ji} \leq \tau = 0.5\), cf. Remark 6.

4. COMPARISON OF LINEAR MAS WITH DELAYS AND ANALYTICAL CONSENSUS CONDITION FOR FIRST ORDER MAS

With Theorem 5, we have a simple tool to compare different MAS with delay. To this end, we consider
\[
\bar{G}(j\omega) = \frac{(j\omega)^p + \sum_{k=0}^{p-1} \alpha_k(j\omega)^k}{\sum_{k=0}^{p-1} \alpha_k(j\omega)^k} \notin \Omega_i(\omega),
\] (13)
with \(i = 1, 2, 3\). It can be seen in Figures 2 to 4 that \((\Omega_2(\omega) \cap \mathbb{C}_+) \subset \Omega_3(\omega)\), where \(\mathbb{C}_+\) is the closed upper half plane, i.e. the subset of \(\Omega_2\) with positive imaginary part is contained in \(\Omega_3\). Moreover, \(\Omega_2(\omega) \subset \Omega_1(\omega)\) and \(\Omega_3(\omega) \subset \Omega_1(\omega)\). Hence, if \(\exists \{\bar{G}(j\omega)\} > 0\) for \(\omega > 0\), then the consensus condition for MAS without self-delay is less restrictive than the consensus conditions for MAS with identical self-delay. And both are less restrictive than the consensus condition for MAS with different delays.

The new set-valued consensus condition is now transformed into an analytical consensus condition for a single integrator MAS
\[
\dot{x}_i(t) = -K \sum_{j=1}^{N} a_{ji} (x_i(t - T_{ji}) - x_j(t - \tau_{ji})),
\] (14)
where \(K > 0\) is the coupling gain. We are interested in the largest coupling gain \(K\) that guarantees consensus for a given delay bound \(\tau\). The undelayed counterpart of this MAS is the standard model in the literature on consensus and achieves consensus as long as the graph is connected for arbitrary large \(K\), e.g., Olafsson-Saber et al. (2007). We have shown in Münz et al. (2009b) that first order MAS without self-delay, i.e. \(T_{ji} = 0, \forall i, j\), achieve consensus for all \(K > 0\) and first order MAS with identical self-delays, i.e. \(T_{ji} = \tau_{ji}, \forall i, j\), reach consensus for \(K < \frac{\tau}{2}\).

Corollary 7. A first order MAS (14) with parameter \(K > 0\), arbitrary size \(N \in \mathbb{N}\), arbitrary delays \(T_{ji} \leq \tau, \tau_{ji} \leq \tau\), and arbitrary connected topology achieves consensus asymptotically if
\[ K < \frac{\pi}{4\tau} \]  

(15)

The proof is given in Section 6.

Corollary 7 gives a simple and accurate consensus condition for single integrator MAS. This shows the applicability of the set-valued condition in Theorem 5. Surprisingly, this condition is exactly the same as for identical self-delays, i.e. \( T_{ij} = \tau_{ij} \), cf. Münz et al. (2009b); Lestas and Vinnicombe (2007b); Bliman and Ferrari-Trecate (2008), and if all delays are identical, i.e. \( T_{ij} = \tau_{ij} = \tau \), cf. Olfati-Saber and Murray (2004).

5. CONCLUSIONS

We provided graphical consensus conditions for linear MAS with undirected communication networks with symmetric delay-delays, considering independent delays for the agent’s own state and its neighbor’s states. We have compared this result to previous publications that consider MAS without self-delay and MAS with identical self-delay. Moreover, we derived analytical necessary and sufficient conditions for first order MAS to reach consensus. This result revealed that the maximal allowable, delay-dependent gain for first order MAS with different delays is exactly the same as if all delays are identical. Summarizing, the generalized Nyquist criterion shows to be a very suitable tool to prove consensus in linear MAS with different kind of heterogeneous delayed feedback.

6. TECHNICAL PROOFS

For the proof of Lemma 4, we will use the following result which follows directly from the definition of the eigenvalues:

Lemma 8. Given a diagonal positive definite matrix \( D \in \mathbb{R}^{n \times n} \) and a matrix \( M \in \mathbb{C}^{n \times n} \), the following holds

\[ \sigma(D^{-1}M) = \sigma(D^{-\frac{1}{2}}MD^{-\frac{1}{2}}). \]

(16)

Note that the spectrum of \( D^{-1}M \) is real if \( M \) is real symmetric because, in this case, \( D^{-\frac{1}{2}}MD^{-\frac{1}{2}} \) is real symmetric.

Proof. [of Lemma 4] The proof is based on the field of values of a matrix, see e.g. Horn and Johnson (1991). The field of values \( F \) of a matrix \( M \in \mathbb{C}^{n \times n} \) is defined as \( F(M) = \{ v^\ast M v : v \in \mathbb{C}^n, v^\ast v = 1 \} \). The most important property of the field of values is that it contains the spectrum \( \sigma \) of \( M \) (the set of eigenvalues of \( M \)), i.e. \( \sigma(M) \subset F(M) \). By Lemma 8, we know that

\[ \sigma(D^{-1}(D\tau(s) - A\tau(s))) = \sigma(D^{-\frac{1}{2}}(D\tau(s) - A\tau(s))D^{-\frac{1}{2}}). \]

Hence, the spectrum of \( D^{-1}(D\tau(j\omega) - A\tau(j\omega)) \) is contained in

\[ F \left( D^{-\frac{1}{2}}(D\tau(j\omega) - A\tau(j\omega))D^{-\frac{1}{2}} \right) = F \left( D^{-\frac{1}{2}}D\tau(j\omega)D^{-\frac{1}{2}} + F \left( D^{-\frac{1}{2}}A\tau(j\omega)D^{-\frac{1}{2}} \right) \right), \]

(17)

where \( F \left( D^{-\frac{1}{2}}D\tau(j\omega)D^{-\frac{1}{2}} \right) = \text{Co} \{ e^{-j\varphi} : \varphi \in [0, \omega \tau] \}. \) Moreover, we have for \( v = 1 \)

\[ v^\ast D^{-\frac{1}{2}}A\tau(s)D^{-\frac{1}{2}}v = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e^{-s_{ij}} \left( \frac{v_i^\ast v_j}{d_i} + \frac{v_j^\ast v_i}{d_i} \right) \]

\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e^{-s_{ij}} \left( \frac{v_i^\ast v_j}{d_i} + \frac{v_j^\ast v_i}{d_i} \right) \]

\[ = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e^{-s_{ij}} \beta_{ij} \frac{|v_i|^2}{d_i} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} e^{-s_{ij}} \beta_{ij} \frac{|v_i|^2}{d_i}, \]

for some \( \beta_{ij} \in [-1, 1] \) and some \( \beta_{ij} \in [-1, 1] \). Thus,

\[ F \left( D^{-\frac{1}{2}}A\tau(j\omega)D^{-\frac{1}{2}} \right) \subset \text{Co} \left\{ e^{-j\varphi} : \varphi \in [0, \omega \tau] \right\}. \]

Finally, we obtain (9) with (17) and \( \sigma(I - \gamma) = 1 - \sigma(\gamma). \)

Proof. [of Theorem 5] Using Lemma 4, we transform the Nyquist criterion as follows

\[ 1 \notin \text{Co} \{ \sigma(G(j\omega)), 0 \} \leq \frac{\sum_{k=0}^{p} \alpha_k (j\omega)^k}{\sum_{k=0}^{p} \alpha_k (j\omega)^k} \Omega_1(\omega), \]

i.e. (12) guarantees that the eigenloci of the return ratio \( G \) neither touch nor encircle the point +1 for \( \omega \neq 0 \). For \( \omega = 0 \), we have \( I - D^{-1}(D\tau(0) - A\tau(0)) = D^{-1}A \). Since the graph is connected, all eigenvalues of \( D^{-1}A \) are in the set \([-1, 1] \) and there is a single eigenvalue at +1. The corresponding eigenvector is 1. This eigenvalue and eigenvector correspond to the consensus subspace (2). Since all eigenloci satisfy the conditions of Theorem 3, we conclude that consensus is reached asymptotically.

Proof. [of Corollary 7] Consider (12) and note that consensus is achieved if \( 1 + j\frac{d}{K} \notin \Omega_1(\omega) \). This is illustrated in Figure 6. Note that \( \Omega_1(\omega) \) always contains \( z \in \mathbb{C} \) with \( \Re(z) = 1 \) and \( \Im(z) > 0 \). The imaginary part of these \( z \) is bounded by

![Figure 6. Illustration of 1 + j\omega/\kappa and \Omega_1(\omega)](image-url)
\[ \Im(z) \leq \sin(\omega \tau) + (1 - \cos(\omega \tau)) \tan\left( \frac{\omega \tau}{2} \right) \]
\[ = \sin(\omega \tau) \left( 1 + \frac{1 - \cos \omega \tau}{1 + \cos \omega \tau} \right). \]

Hence, \( 1 + \frac{j \omega}{\tau} \notin \Omega_1(\omega) \) if \( \frac{\omega}{\tau} > \Im(\omega \tau) \) for \( \omega \tau < \frac{\pi}{2} \). We reformulate this inequality as
\[ 1 + \frac{j \omega}{\tau} > \sin(\omega \tau) \left( 1 + \frac{1 - \cos \omega \tau}{1 + \cos \omega \tau} \right) = f(\omega \tau). \]

Plotting \( f \) for \( \omega \tau \in [0, \frac{\pi}{2}] \) shows that \( f \) is strictly increasing in this interval. Hence, \( \frac{1}{\omega \tau} > f\left(\frac{\omega}{\tau}\right) = \frac{1}{\omega}, \) i.e. (15), guarantees consensus.

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