Model predictive control of constrained non-linear time-delay systems

MARCUS REBLE*
Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany
*Corresponding author: reble@ist.uni-stuttgart.de

REZA MAHBOOBI ESFANJANI AND SEYYED KAMALEDIN Y. NIKRAVESH
Electrical Engineering Department, Amirkabir University of Technology, 424 Hafez Avenue, Tehran, Iran

AND

FRANK ALLGÖWER
Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany

[Received on 29 January 2010; revised on 30 August 2010; accepted on 15 November 2010]

This paper proposes a model predictive control scheme for non-linear time-delay systems with input constraints. Based on the results for systems without delays, asymptotic stability of the closed loop is guaranteed by utilizing an appropriate terminal cost functional and an appropriate terminal region such that the optimal cost for the finite-horizon problem is an upper bound on the optimal cost for the associated infinite-horizon problem. Two structured procedures are presented to determine offline the terminal cost and the terminal region for a class of non-linear time-delay systems. For both procedures, sufficient conditions can be formulated in terms of linear matrix inequalities based on the Jacobi linearization of the system about the origin. The first procedure uses a combination of Lyapunov–Krasovskii and Lyapunov–Razumikhin conditions in order to compute a locally stabilizing controller and a control invariant region. The second procedure only applies Lyapunov–Krasovskii arguments but may yield more complicated control invariant regions. The effectiveness of both schemes is compared for the example of a continuous stirred tank reactor with recycle stream.

Keywords: model predictive control; time delay; non-linear control; Lyapunov–Krasovskii; Lyapunov–Razumikhin.

1. Introduction

The dynamic models of many technical, biological and economical systems involve both non-linearities and time delays in the states, especially systems that include transportation of material or data. Therefore, considerable attention has been recently devoted to the control of non-linear time-delay systems (Richard, 2003; Jankovic, 2001, 2003, 2005; Márquez-Martínez & Moog, 2004; Papachristodoulou, 2005). However, there are only few results regarding the control of non-linear time-delay systems with input constraints.

Model predictive control (MPC), also known as receding horizon control, is one of the few control strategies for non-linear systems which is capable to handle constraints. However, MPC using
finite-horizon optimal control problems without additional stabilizing constraints does not guarantee closed-loop stability in general, see e.g. Raff et al. (2006) for a practical example of this effect. Therefore, several approaches have been developed to avoid this problem for delay-free systems, e.g. Chen & Allgöwer (1998), Mayne et al. (2000), Fontes (2000) and Jadabaie et al. (2001). In most cases, closed-loop stability is guaranteed by using an appropriately chosen terminal cost and terminal region.

For linear time-delay systems, there exist several schemes for MPC, e.g. Kwon et al. (2003, 2004), Han et al. (2008) and Shi et al. (2009). In contrast, there are only few results concerning MPC for non-linear time-delay systems. For a special class of non-linear time-delay systems, Kwon et al. (2001a,b) present an MPC controller with guaranteed closed-loop stability that is based on a suitably defined terminal cost functional. However, a globally stabilizing control law has to exist in order to calculate the terminal cost. Furthermore, simple conditions for the stabilizing control law can only be derived if the non-linear system is linearly bounded in the delayed state, which renders this approach very restrictive.

The scheme proposed by Mahboobi Esfanjani & Nikravesh (2009) uses control Lyapunov–Krasovskii functionals in order to guarantee stability without terminal constraints. However, no input constraints are considered. In Raff et al. (2007), an expanded zero-terminal state constraint is used to assure stability of the closed loop. The resulting optimal control problem is rather difficult to solve because the system has to be steered to the origin in finite time. This leads to feasibility problems especially for short prediction horizons. Furthermore, an exact satisfaction of a zero-terminal state constraint does require an infinite number of iterations in the optimization. This makes the approach unattractive from a computational point of view.

In this contribution, an MPC scheme for non-linear time-delay systems with input constraints is presented. A suitable finite terminal region and terminal cost functional are used to guarantee asymptotic stability of the closed loop. Two procedures are presented for calculating the stabilizing design parameters based on the linearization about the origin. In contrast to the delay-free case, the terminal region for non-linear time-delay systems cannot be defined as a sublevel set of a quadratic Lyapunov–Krasovskii functional of the linearized system. In deed, additional arguments are necessary to derive the terminal region based on the linearization. In the first procedure, based on the results of Mahboobi Esfanjani et al. (2009), these additional arguments are Lyapunov–Razumikhin conditions on the local control law. The second procedure only uses standard Lyapunov–Krasovskii conditions on the local control law and is therefore less restrictive. However, a more complicated terminal region is obtained.

The remainder of this paper is organized as follows. In Section 2, the problem set-up considered in this work is presented. In Section 3, the MPC set-up for non-linear time-delay systems is described and sufficient conditions for asymptotic stability are stated. In Section 4, general remarks and preliminary calculations for calculating appropriate terminal regions and terminal cost functionals are given. A procedure for calculating those design parameters is explained in Section 5 using a combination of Lyapunov–Krasovskii and Lyapunov–Razumikhin arguments. Section 6 presents another procedure using only Lyapunov–Krasovskii arguments. Simulation results of a continuous stirred tank reactor (CSRT) with recycle stream are provided in Section 7. Summary and concluding remarks are given in Section 8.

**Notation.** Let \( \mathbb{R}^+ \) denote the non-negative real numbers and \( \mathbb{R}^n \) denote the \( n \)-dimensional Euclidean space with the standard norm \( | \cdot | \). \( \| P \| \) is the induced 2-norm of matrix \( P \). Given \( \tau > 0 \), let \( \mathcal{C}_\tau = \mathcal{C}([-\tau, 0], \mathbb{R}^n) \) denote the Banach space of continuous functions mapping the interval \([-\tau, 0]\) into \( \mathbb{R}^n \) with the topology of uniform convergence. The norm on \( \mathcal{C}_\tau \) is defined as \( \| \varphi \|_\tau = \sup_{-\tau \leq s \leq 0} | \varphi(s) | \). A segment \( x_t \in \mathcal{C}_\tau \) is defined by \( x_t(s) = x(t + s), s \in [-\tau, 0] \). \( \lambda_{\max}(P) \) and \( \lambda_{\min}(P) \) refer to the maximal and minimal eigenvalue of matrix \( P \), respectively. A function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is said to belong to class \( \mathcal{K}_\infty \) if it is continuous, strictly increasing, \( f(0) = 0 \) and \( f(s) \rightarrow \infty \) as \( s \rightarrow \infty \).
2. Problem set-up

Consider the non-linear time-delay system

\begin{align}
\dot{x}(t) &= f(x(t), x(t-\tau), u(t)), \\
x(\theta) &= \varphi(\theta), \quad \forall \theta \in [-\tau, 0]
\end{align}

(2.1a, 2.1b)

in which \(x(t) \in \mathbb{R}^n\) is the instantaneous state, \(u(t) \in \mathbb{R}^m\) is the control input subject to input constraints \(u(t) \in \mathcal{U}\) and \(\varphi \in \mathcal{C}_\tau\) is the initial function. The time delay \(\tau\) is assumed to be known and constant. The function \(f\) is continuously differentiable. The constraint set \(\mathcal{U} \subset \mathbb{R}^m\) is compact, convex, and contains the origin in its interior. Without loss of generality, we assume \(x_t = 0\) to be an equilibrium of system (2.1) for \(u = 0\), i.e. \(f(0, 0, 0) = 0\). The problem of interest is to stabilize \(x_t = 0\) and to achieve some optimal performance via MPC.

3. MPC for non-linear time-delay systems

MPC is formulated as solving online a finite-horizon optimal control problem. Based on the measurements obtained at time \(t\), the controller predicts the future behaviour of the system over a finite prediction horizon \(\mathcal{I}\) and determines the control input such that a predetermined cost functional \(J\) is minimized. In order to incorporate a feedback mechanism, the obtained open-loop solution to this optimal control problem will be implemented only until the next measurement becomes available. Based on the new measurement, the solution of the optimal control problem is repeated for a now shifted horizon and again implemented until the next sampling instant.

It is well known that an inappropriate definition of the finite-horizon optimal control problem may cause instability especially if the horizon is too short, see e.g. the practical example in Raff et al. (2006). To guarantee closed-loop stability, certain conditions in the finite-horizon optimization problem have to be met in order to assure that the associated optimal cost can be used as a Lyapunov function for the closed-loop control system (Mayne et al., 2000).

The MPC set-up considered in this work is closely related to the classical schemes for delay-free systems (Chen & Allgöwer, 1998; Mayne et al., 2000). A locally asymptotically stabilizing control law is designed in some neighbourhood \(\Omega \subseteq \mathcal{C}_\tau\) of the equilibrium. With this locally stabilizing controller, an upper bound on the infinite-horizon cost is computed and used as a terminal cost. Furthermore, a constraint is added to the open-loop optimal control problem that requires the final state \(x_t\) to lie within the terminal region \(\Omega\).

The open-loop finite-horizon optimal control problem at time \(t\) with prediction horizon \(\mathcal{I}\) is formulated as

\[
\min_{u(t)} J(x_t, u; t, \mathcal{I}) = \int_t^{t+\mathcal{I}} F(x(t'), u(t'))dt' + E(x_{t+\mathcal{I}})
\]

(3.1)

subject to

\begin{align}
\dot{x}(t') &= f(x(t'), x(t' - \tau), u(t')) , \\
u(t') &\in \mathcal{U} , \\
x_{t+\mathcal{I}} &\in \Omega \subseteq \mathcal{C}_\tau
\end{align}

(3.2)
in which \( x(t') \) is the predicted trajectory starting from initial condition \( x_t = x(t + \theta), \, -\tau \leq \theta \leq 0 \), and driven by \( u(t') \) for \( t' \in [t, t + \Xi] \). The terminal region \( \Omega \) is a closed set and contains \( 0 \in C_t \) in its interior. \( E: C_t \rightarrow \mathbb{R}^+ \) is a suitably defined positive-definite terminal cost functional for which a class \( \mathcal{X}_\infty \) function \( \hat{E} \) exists such that \( E(x_t) \geq \hat{E}(|x(t)|) \). The stage cost \( F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^+ \) is continuous, \( F(0, 0) = 0 \) and there is a class \( \mathcal{X}_\infty \) function \( \hat{F}: \mathbb{R} \rightarrow \mathbb{R}^+ \) such that

\[
F(x, u) \geq \hat{F}(|x|) \quad \text{for all } x \in \mathbb{R}^n, \, u \in \mathbb{R}^m.
\]

Examples for such a terminal region and a terminal cost functional will be derived in Sections 5 and 6. We assume that the optimal control which minimizes \( J(x_t, u; t, \Theta) \) exists and is given by \( u^*(t'; x_t, t) \), \( t' \in [t, t + \Xi] \). The associated optimal cost is then denoted by \( J^*(x_t; t, \Theta) \). The control input to the system is defined by the optimal solution of problem (3.1), (3.2) at sampling instants \( t_i = i \Delta \) in the usual receding horizon fashion

\[
u(t) = u^*(t; x_t, t_i), \quad t_i \leq t \leq t_i + \Delta.
\]

The implicit feedback controller resulting from application of (3.4) is asymptotically stabilizing provided the following conditions are satisfied.

**Assumption 3.1** The open-loop finite-horizon problem (3.1), (3.2) admits a feasible solution at the initial time \( t = 0 \).

**Assumption 3.2** For the non-linear time-delay system (2.1), there exists a locally asymptotically stabilizing controller \( u(t) = k(x_t) \in \mathcal{U} \), such that the terminal region \( \Omega \) is controlled positively invariant and

\[
\forall x_t \in \Omega: \hat{E}(x_t) \leq -F(x(t), k(x_t)). \tag{3.5}
\]

We can summarize the main result regarding asymptotic stability of the closed-loop system as follows.

**Theorem 3.1** Assume that in the finite-horizon optimal control problem (3.1), (3.2) the design parameters—the stage cost \( F \), the terminal cost functional \( E \), the terminal region \( \Omega \) and the prediction horizon \( \Xi \)—are selected such that Assumptions 3.1 and 3.2 are satisfied. Then, the closed-loop system resulting from the application of the model predictive controller (3.4) to system (2.1) is asymptotically stable. The region of attraction is the set of all initial conditions for which problem (3.1), (3.2) is initially feasible.

**Proof.** The proof is given in Appendix A. A different version of the proof can be found in Mahboobi Esfanjani et al. (2009). \( \square \)

Note that the existing MPC schemes for non-linear time-delay systems by Kwon et al. (2001a,b) and Raff et al. (2007) can be viewed as special cases of this general result. In the next sections, two schemes for calculating appropriate design parameters, i.e. terminal region \( \Omega \) and terminal cost functional \( E \), are derived for non-linear time-delay systems.

### 4. Calculation of the terminal region and terminal cost

The key element in the stabilizing MPC scheme above are a suitably determined terminal cost \( E \) and terminal region \( \Omega \). The goal of the following sections is to derive two simple procedures for calculating an appropriate terminal cost and terminal region. In both cases, linear matrix inequality (LMI) conditions
are given which guarantee satisfaction of Assumption 3.2 and, thus, closed-loop stability with the MPC scheme presented.

Note that in the delay-free case, the terminal region is mostly defined as a sublevel set of the terminal cost function and therefore invariance is trivially guaranteed by the second condition $\dot{E} \leq -F$. Although in principle, it is also possible for time-delay systems to use level sets, it is not possible to derive a simple procedure along the lines of Chen & Allgöwer (1998) for the calculation of suitable cost functionals for non-linear time-delay systems based on the Jacobi linearization about the origin. The reason for this is as follows. In the delay-free case, it is possible to determine a sufficiently small sublevel set of the positive-definite Lyapunov function of the linearized system such that the non-linear terms are sufficiently small compared to the linear terms for all states in this set. Thus, a sufficiently small sublevel set is positively invariant also for the non-linear system. However, for the infinite-dimensional case, the non-linear terms might not be small compared to the linear terms even in an arbitrarily small sublevel set of a positive-definite Lyapunov–Krasovskii functional of the linearized system. Thus, even an arbitrarily small sublevel set might not be positively invariant for the non-linear system. Roughly speaking, this is possible because even for arbitrarily small values of a positive-definite functional, the norm of its argument might be arbitrarily large. This is similar to the well-known fact that there is no equivalence between different norms in infinite-dimensional spaces. Due to this reason, additional arguments have to be used for infinite-dimensional systems.

Two special cases can be detected for which Assumption 3.2 is directly satisfied. First, the work of Raff et al. (2007) considers an expanded zero-terminal state constraint. Thus, the terminal region only consists of one single point in the function space, i.e. the steady state itself. This approach is unattractive from a computational point of view because of the aforementioned reasons. Second, the work in Kwon et al. (2001a) uses the whole state space as terminal region and requires the knowledge of a globally stabilizing controller which might not be easily computed especially for the case of input constraints. In both cases, the invariance of the terminal region is trivially satisfied.

In the following, two procedures to calculate finite terminal regions are presented. Both procedures use the Jacobi linearization of the system about the origin. However, two different types of terminal regions are obtained. In Section 5, a combination of Lyapunov–Krasovskii and Lyapunov–Razumikhin arguments is used to calculate a local control law such that a simple terminal region is controlled positively invariant. In Section 6, it is shown that a Lyapunov–Krasovskii condition on the local control law is sufficient if a more complicated terminal region is considered. Since both schemes use the Jacobi linearization of the non-linear system about the origin, some preliminary calculations and definitions are stated in the next section.

4.1 Preliminaries

In order to obtain the local control law, we consider the Jacobi linearization

$$\dot{x}(t) = \hat{f}(\bar{x}(t), \bar{x}(t-\tau), u(t)) = A \bar{x}(t) + A_r \bar{x}(t-\tau) + Bu(t),$$

(4.1)
of system (2.1) about the origin with matrices

$$A = \left. \frac{\partial f}{\partial x(t)} \right|_{x_2=0, u=0}, \quad A_r = \left. \frac{\partial f}{\partial x(t-\tau)} \right|_{x_2=0, u=0}, \quad B = \left. \frac{\partial f}{\partial u(t)} \right|_{x_2=0, u=0}. \quad (4.2)$$

Define $\Phi$ as the difference between the non-linear system (2.1) and its Jacobi linearization (4.1)

$$\Phi(x(t), x(t-\tau), u(t)) = f(x(t), x(t-\tau), u(t)) - \hat{f}(x(t), x(t-\tau), u(t))$$

$$= f(x(t), x(t-\tau), u(t)) - [Ax(t) + A_r x(t-\tau) + Bu(t)]. \quad (4.3)$$
For the sake of brevity, \( \Phi(x, u(t)) \) will be used as a short hand for \( \Phi(x(t), x(t - \tau), u) \). Since \( f \) is continuously differentiable and \( \Phi \) only consists of higher order terms, i.e. it contains no linear terms, for any \( \gamma > 0 \), there exists a \( \delta = \delta(\gamma) \) such that

\[
|\Phi(x, Kx(t))| < \gamma |(x^T(t), x^T(t - \tau))|^T \quad \text{for all } |(x^T(t), x^T(t - \tau))|^T < \delta
\]

and any local control law \( u(t) = Kx(t) \).

5. Stabilizing design parameters based on Lyapunov–Krasovskii and Lyapunov–Razumikhin arguments

In this section terminal regions of the form

\[
\Omega_\alpha = \{ x_t: \max_{\theta \in [-\tau, 0]} x(t + \theta)^T P x(t + \theta) \leq \alpha \}
\]

(5.1)

are considered. Razumikhin-type arguments are used in Section 5.1 in order to ensure the controlled positive invariance of such a terminal region for some \( \alpha > 0 \) and some symmetric positive-definite matrix \( P \). To this end, a locally stabilizing linear control law \( u(t) = Kx(t) \) is derived which renders the terminal region \( \Omega_\alpha \) positively invariant with respect to system (2.1) and satisfies the input constraints. In the next step, a Lyapunov–Krasovskii functional \( E(x_t) \) is used in Section 5.2 as an upper bound of the infinite-horizon cost for a stage cost of form

\[
F(x(t), u(t)) = x(t)^T Q x(t) + u(t)^T R u(t),
\]

(5.2)

with symmetric positive-definite matrices \( Q \) and \( R \). LMI conditions are given such that the condition \( \dot{E}(x_t) \leq -F(x(t), Kx(t)) \) holds for all states inside the terminal region when using the local control law. Combining both arguments, it directly follows that Assumption 3.2 is satisfied and asymptotic stability of the closed loop can be guaranteed.

5.1 Invariance of the terminal region

In this part, we derive conditions in terms of LMIs for the linear control law to render a terminal region \( \Omega_\alpha \) of form (5.1) positively invariant.

**Lemma 5.1** Consider system (2.1). If there exist symmetric matrices \( A > 0, A_i > 0, i = 1, 2, 3 \) and a matrix \( \Gamma \) of appropriate dimensions solving the following LMIs:

\[
\begin{bmatrix}
\Xi_1 + 2\tau A & \tau A \Gamma (A A + B \Gamma) & \tau A_i A & \tau A_i A \\
* & -\tau A_1 & 0 & 0 \\
* & * & -\tau A_2 & 0 \\
* & * & * & -\tau A_3
\end{bmatrix} < 0,
\]

(5.3)

in which \( \Xi_1 = A(A + A)^T + (A + A \Gamma) A + \Gamma^T B^T + B \Gamma \), then the local control law \( u(t) = Kx(t) \) with \( K = \Gamma A^{-1} \) renders the region \( \Omega_\alpha \) in (5.1) positively invariant for \( P = A^{-1} \) and some \( \alpha > 0 \).

**Proof.** The proof is given in Appendix B and contains an implicit formula for \( \alpha \) given by (B.14). \( \square \)
5.2 Terminal cost functional

In this subsection, we derive a simple condition for the choice of a terminal cost functional such that Condition (3.5) is satisfied for the local control law \( u(t) = K x(t) \). To this end, we consider a simple functional of the form

\[
E(x_t) = x(t)^T P x(t) + \int_{-\tau}^{0} x^T(t + \theta) S x(t + \theta) d\theta
\]

(5.5)

in which \( P \) and \( S \) are \( n \times n \) symmetric positive-definite constant matrices.

**Lemma 5.2** Consider system (2.1) and stage cost \( F \) in (5.2). If there exist symmetric matrices \( A > 0 \) and \( \Upsilon > 0 \), a matrix \( \Gamma \) and a constant positive scalar \( \epsilon \) solving the following LMI:

\[
\begin{bmatrix}
\Xi_2 + \Upsilon + \epsilon I & A^T A & A Q^{1/2} & \Gamma^T R^{1/2} \\
* & -\Upsilon + \epsilon I & 0 & 0 \\
* & * & -I & 0 \\
* & * & * & -I
\end{bmatrix} < 0
\]

(5.6)

in which \( \Xi_2 = A A^T + A A + \Gamma^T B^T + B \Gamma \), then the control law \( u(t) = K x(t) \) with \( K = \Gamma A^{-1} \) ensures \( \dot{E}(x_t) \leq -F(x_t, K x(t)) \) for all \( x_t \in \Omega_\alpha \) for some \( \alpha > 0 \). The cost functional \( E \) is given by (5.5) with parameters \( P = A^{-1} \) and \( S = A^{-1} \Upsilon A^{-1} \).

**Proof.** The proof is given in Appendix B and contains an implicit conditions on \( \alpha \) given by (B.17). \( \square \)

5.3 Combination of both results

Combining the previous results from Sections 5.1 and 5.2, we directly obtain the following theorem.

**Theorem 5.1** If there exist symmetric positive-definite matrices \( A, A_1, A_2, A_3, \Upsilon \) and a matrix \( \Gamma \) such that LMIs (5.3), (5.4) and (5.6) admit a feasible solution, then there exists \( \alpha > 0 \) small enough such that the terminal region \( \Omega_\alpha \) in (5.1) and terminal cost functional \( E \) in (5.5) with \( P = A^{-1} \) and \( S = A^{-1} \Upsilon A^{-1} \) satisfy Assumption 3.2 for stage cost \( F \) in (5.2).

**Proof.** The proof directly follows from Lemmas 5.1 and 5.2 and because it is always possible to choose \( \alpha \) small enough such that the input constraints are satisfied for all \( x_t \in \Omega_\alpha \). \( \square \)

6. Stabilizing design parameters based on Lyapunov–Krasovskii arguments

In this section, another procedure to obtain stabilizing design parameters is proposed. It is shown that a suitable terminal region can be calculated based on the Jacobi linearization without requiring a Razumikhin-condition on the local control law. However, the resulting terminal region is of more complicated form than the one used in Section 5 and is described as the intersection of a sublevel set of the Lyapunov–Krasovskii functional of the linearized system and a set defined by a bound on the norm.

**Theorem 6.1** Consider the non-linear time-delay system (2.1) and quadratic stage cost

\[
F(x(t), u(t)) = x(t)^T Q x(t) + u(t)^T R u(t)
\]
with symmetric positive-definite matrices $Q$ and $R$. If there exist symmetric matrices $A > 0$ and $Y > 0$, a matrix $T$ and a constant positive scalar $\varepsilon > 0$ solving the following LMI:

$$
\begin{bmatrix}
\Xi_2 + Y + \varepsilon I & A_T A & A Q^{1/2} T R^{1/2} \\
* & -Y + \varepsilon I & 0 \\
* & * & -T & 0 \\
* & * & * & -I \\
\end{bmatrix} < 0
$$

(6.1)

in which $\Xi_2 = A A^T + A A + T B^T + B T$ and $A$, $A_T$ and $B$ result from the Jacobi linearization about the origin (4.2), then the control law $u(t) = K x(t)$ with $K = T^{-1}$ locally asymptotically stabilizes the non-linear time-delay system (2.1). Furthermore, consider the cost functional $E$ given by

$$
E(x_t) = x(t)^T P x(t) + \int_{-\tau}^{0} x^T(t + \theta) S x(t + \theta) d\theta
$$

(6.2)

with parameters $P = A^{-1}$ and $S = A^{-1} Y A^{-1}$ and the terminal region defined by

$$
\Omega = \left\{ x_t : E(x_t) \leq \frac{\lambda_{\min}(P)}{4} \delta(\gamma)^2, \|x_t\|_\tau \leq \frac{\delta(\gamma)}{2} \right\},
$$

(6.3)

with $\gamma > 0$ chosen small enough such that

$$
\gamma \leq \frac{\lambda_{\min}(P)}{2 \lambda_{\max}(P)}
$$

(6.4)

and small enough such that $|x| < \frac{\delta(\gamma)}{2} \Rightarrow u = K x \in U$. Then the local control law ensures that

(a) for all $x_t \in \Omega$ the condition $\dot{E}(x_t) \leq -F(x(t), K x(t))$ is satisfied and

(b) the terminal region $\Omega$ is positively invariant.

Proof. The proof is given in Appendix C and in Reble & Allgöwer (2010). \qed

From Theorem 3.1 asymptotic stability directly follows for the closed loop using the MPC controller (3.4) with terminal cost $E$ in (6.2) and terminal region $\Omega$ in (6.3).

Note that similar to the results in Section 5, the terminal region is not defined as a sublevel set of the corresponding terminal cost functional. Instead, it is defined as the intersection of a sublevel set and the set of all functions in $\mathcal{C}_{\tau}$ with norm $\|\cdot\|_\tau$ less than some small positive number. Note that this is not necessary in the finite-dimensional case, i.e. for systems without time delay because for given positive-definite function $E(x) : \mathbb{R}^n \to \mathbb{R}^+$ and positive $\delta > 0$, it is always possible to find a small positive constant $\varepsilon > 0$ such that for all $x$ in the $\varepsilon$-level set of $E$ it holds that $|x| < \delta$ for any vector norm $|\cdot|$. However, this relation between norms does not hold in the infinite-dimensional case. This makes the use of more complicated arguments necessary.

Furthermore, the result in Theorem 6.1 is less conservative than the one obtained in Section 5 in the sense that if there is a solution to the conditions in Mahboobi Esfanjani et al. (2009), then the assumptions in Theorem 6.1 are satisfied. This can be easily seen as the LMI condition in Theorem 6.1 directly relates to the Lyapunov-Krasovskii condition for the terminal cost functional in Lemma 5.2. However, the terminal region calculated in this work is more complicated and might make the numerical solution of the open-loop optimal control problem more difficult.
7. Numerical example: (CSRT) with recycle stream

In this section, simulation results for the model of a (CSRT) with recycle stream are provided. The model and parameters are taken from the example of Findeisen & Allgöwer (2000) and extended with a recycle stream. The equations of the reactor following from the mass and energy balance are given by

\[
\begin{align*}
\dot{c}(t) &= a_1(c_{\text{in}}(t) - c(t)) - 2K(T(t))c(t)^2, \\
\dot{T}(t) &= a_1(T_{\text{in}}(t) - T(t)) + a_2(T_k(t) - T(t)) + a_3K(T(t))c(t)^2
\end{align*}
\]

in which

\[
c_{\text{in}}(t) = (1 - \nu)c_f + \nu c(t - \tau), \quad T_{\text{in}}(t) = (1 - \nu)T_f + \nu T(t - \tau).
\]

\(T\) and \(c\) denote the temperature and concentration of the reactant inside the reactor. \(T_f\) and \(c_f\) are the temperature and concentration of the inflow and both assumed to be constant. The manipulated input is the heating jacket temperature \(T_k\). The coefficient \(\nu\) is the recirculation coefficient with \(\nu = 0.5\) in \([0, 1]\) and \(\tau = 20\) s the recycle time. The simulation parameters are chosen as follows. The rate of reaction \(K(T)\) is given by the Arrhenius law \(K(T) = k_0 e^{-\frac{a_4}{T}}\). The other model parameters are

\[
a_1 = \frac{q}{V}, \quad a_2 = \frac{k_w F_k}{\rho c_p V}, \quad a_3 = \frac{-\Delta h_r}{\rho c_p}, \quad a_4 = \frac{E_A}{R}
\]

with

\[
q = 0.1 \text{ l/min}, \quad V = 11, \quad k_w = 0.1 \text{ cal cm}^2\text{min K}^{-1}, \quad F_k = 250\text{cm}^2, \quad \rho c_p = 659 \text{ cal K}^{-1}
\]

\[
\Delta h_r = -20000 \text{ cal mol}^{-1}, \quad E_A = -\Delta h_r, \quad R = 1.9864 \text{ cal mol}^{-1} K^{-1}, \quad k_0 = 33 \times 10^9 \text{ l mol min}^{-1}
\]

The goal is to stabilize the unstable steady state \(T_s = 345\) K, \(cT_s = 4.24\) mol/l for constant inflow parameters \(T_f = 290\) K, \(c_f = 6.67\) mol/l and heating jacket temperature \(T_{k,s} = 389\) K. The input \(T_k\) is constrained between 349 K and 429 K, i.e. \(|u| = |T_k - T_{k,s}| \leq 40\) K.

In order to apply the results of the previous sections, the Jacobi linearization about the steady state is calculated and the resulting LMIs are solved in Matlab using Yalmip (Löfberg, 2004). The weighting matrices are chosen as \(Q = 100 I\) and \(R = I\). The resulting local control law is

\[
T_k = T_{k,s} + K(x(t) - xT_s) = T_{k,s} + \begin{bmatrix}-49.18 & \text{Kl mol}^{-1} \\ -26.41 & \text{mol}^{-1} \end{bmatrix} [c(t) - cT_s, T(t) - T T_s]^T.
\]

The solutions of the LMIs in combination with (B.14), (B.17) and (6.4) give conditions on \(\gamma\) for the first and the second scheme, respectively. For \(\gamma\) determined this way, it remains to calculate \(\alpha\) and \(\delta\), respectively, such that Property (4.4) is satisfied inside the terminal region. One possible approach goes as follows: we know that \(\Phi\) only consists of higher-order terms and does not contain any delayed terms for the example considered in this section. Due to the residual of the Taylor series expansion, we know that \(\Phi(x, Kx) = \frac{1}{2\tau} (\xi x)^T \mathcal{H} \Phi(\xi x) (\xi x)\) for some \(\xi \in [0, 1]\). Here, \(\mathcal{H} \Phi(\xi x)\) denotes the Hessian matrix of \(\Phi\) with respect to \(x\) evaluated at \(\xi x\). By using an upper bound on the Hessian matrix \(\mathcal{H}\), we obtain \(|\Phi(x, Kx)| \leq \frac{1}{2} \mathcal{H} |x|^2\). Now, by choosing \(\alpha\) and \(\delta\) small enough, it is possible to guarantee that \(|x(t)| \leq \frac{2\gamma}{\mathcal{H}}\) for all states \(x_t\) inside the terminal region. Thus, \(|\Phi(x, Kx)| \leq \gamma |x|\) and consequently
Property (4.4) is satisfied for all states inside the terminal region for the desired $\gamma$. Using this approach, the terminal regions for both procedures are calculated as

$$\Omega = \{ x_t : \max_{\theta \in [-\tau, 0]} (x(t + \theta) - x_{ts})^T P (x(t + \theta) - x_{ts}) \leq 0.1 \}, \tag{7.1}$$

$$\Omega = \{ x_t : \| x(t) - x_{ts} \|_\tau \leq 2 \times 10^{-3}, E(x_t) \leq 2 \times 10^{-8} \}. \tag{7.2}$$

The terminal region (7.1) is simpler to use for the numerical calculations than (7.2), however the required Razumikhin condition might be more restrictive for other applications.

The simulation results for a prediction horizon of $\tau = 60$ min are shown in Figs 1 and 2. As can be expected, the model predictive controller stabilizes the unstable steady state $TT_s, cT_s$ while satisfying the input constraints.

In this example, the terminal region is relatively small for both procedures which explains the similar behavior of both controllers in Figs 1 and 2 and shows the still existing conservatism of the proposed scheme. One reason for this is that results based on the Jacobi linearization also lead to conservative results in the delay-free case. This is due to the fact that often only conservative bounds on the non-linearity have to be used such as (4.4), as well as the restriction to quadratic Lyapunov functions which might not be appropriate for non-linear systems. Furthermore, we have only used the simplest quadratic Lyapunov–Krasovskii functional with constant matrices $P$ and $S$ in order to calculate the local control law as a first step. However, it is possible to generalize the principle ideas of using a local control law and either an additional Razumikhin condition or the terminal region defined by the intersection of a sublevel set and a norm-bounded region in $\mathbb{C}_\tau$. For instance, one future step can be the consideration of more complicated functionals, e.g. calculated by means of sum-of-squares techniques as in Papachristodoulou (2005) and Papachristodoulou et al. (2005).

8. Conclusions

In this work, an MPC scheme for non-linear time-delay systems is presented. Asymptotic stability of the closed loop is guaranteed by using an appropriate terminal cost functional and terminal region. Two structured procedures to calculate stabilizing design parameters based on the Jacobi linearization about the steady state are presented. In contrast to the finite-dimensional case, it is not possible to calculate the terminal region as a sublevel set of the Lyapunov functional for the linearized system. Indeed, additional arguments have to be used which can either be Lyapunov–Razumikhin conditions on
the local control law or a suitably adapted terminal region. Exemplarily, simple conditions are given for both procedures in terms of LMIs. Although the simple conditions stated in this work might be restrictive, it is easily possible to generalize the principle ideas of both procedures. For instance, more complicated cost functionals and local controllers can be used, e.g. calculated by means of sum-of-squares techniques or any other technique for linear time-delay systems in order to obtain a suitable terminal region for the non-linear system based on the arguments presented in this work.

Funding

Priority Programme 1305 “Control Theory of Digitally Networked Dynamical Systems” of the German Research Foundation to MR. The authors would like to thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart.

REFERENCES


Appendix A

Proof of Theorem 3.1.

In the following, we show that the proposed predictive controller stabilizes system (2.1) if Assumptions 3.1 and 3.2 are satisfied. The proof follows the lines of Chen (1997), however the infinite-dimensional nature of the state space requires proof of additional properties of the optimal value functional, cf. Lemma A.3, which are directly implied by continuity and positive definiteness in the finite-dimensional case. First, feasibility of the open-loop optimal control problem is addressed in Lemma A.1. To establish asymptotic stability, it will be then shown in Lemma A.2 that the optimal cost $J^*(x_t; t, \Xi)$ of problem (3.1), (3.2) is continuous in the state $x_t$ and is locally lower bounded as shown in Lemma A.3. Continuity of the optimal cost is required for the proof of asymptotic stability as opposed to only convergence. Furthermore, the optimal value functional is not increasing along trajectories of the closed-loop as proven in Lemma A.4. In the last step, asymptotic stability is shown using these results.

Lemma A.1 The open-loop finite-horizon problem (3.1), (3.2) admits a feasible solution for all times $t > 0$, if it is initially feasible.

Proof. Suppose that at time $t$, a feasible solution of (3.1), (3.2) exists, i.e. $u(t'; x_t, t)$, $t' \in [t, t + \Xi]$. At time $t^* \in [t, t + \Delta]$, a feasible, but not necessarily optimal, control input $\hat{u}$ can be constructed by appending control values based on the local controller $k(x_t)$ to the solution at the previous time step $t$

$$\hat{u}(t') = \begin{cases} u^*(t'; x_t, t) & \text{for } t' \in [t^*, t + \Xi], \\ k(x_{t'}) & \text{for } t' \in [t + \Xi, t^* + \Xi]. \end{cases}$$  \hspace{1cm} (A.1)

Hence, $\hat{u}$ consists of two parts: one part of the feasible control that steers the system from $x_t^*$ to $x_{t^*+\Xi} \in \Omega$ inside the terminal region and a second part where the local controller $k(x_t)$ keeps the system trajectory in $\Omega$ for $t + \Xi \leq t' \leq t^* + \Xi$ while respecting the constraints. So, the feasibility of (3.1), (3.2) at time $t$ results in feasibility at time $t^*$ and in particular also at the next sampling instant $t + \Delta$, $\Delta > 0$. By induction (3.1), (3.2) is feasible for every $t > 0$ if it is feasible at initial time $t = 0$. \hfill $\square$

Lemma A.2 The optimal cost $V(x_t) = J^*(x_t; t, \Xi)$ of the open-loop finite-horizon optimal control problem (3.1), (3.2) is continuous in $x_t$ in a neighbourhood of $x_t = 0$ if Assumptions 3.1 and 3.2 are satisfied.

Proof. Let $\varphi \in C_T$ belong to some neighbourhood $C \subseteq C_T$ of the origin and $\varphi \neq 0$. Choose $u = 0$ as a candidate solution to the finite-horizon optimization problem. Now, consider the following system for $t' \in [t, t + \Xi]$

$$\dot{x}(t') = f(x(t'), x(t' - \tau), 0), \quad x_t = \varphi.$$ 

Since $f$ is a continuously differentiable function of its arguments and if $C$ is chosen sufficiently small, a unique solution $x(t)$ exists on $[t, t + \Xi]$ and this solution depends continuously on the initial condition $\varphi$ (see Hale & Lunel, 1993, Theorem 2.2; Kolmanovskii & Myshkis, 1999). Now, let the associated cost functional be denoted by

$$\tilde{J}^*(x_t; t, \Xi) = \int_t^{t+\Xi} F(x(t'), 0)dt' + E(x_{t+\Xi})$$ \hspace{1cm} (A.2)

Since $F$ and $E$ are continuous and $x(t)$ is continuously dependent on $\varphi$ in a neighbourhood of $x_t = 0$, $\tilde{J}^*$ is continuous at $x_t = 0$ in this neighbourhood. Thus, for any $\epsilon > 0$, there exits $\delta(\epsilon)$ such that $\|\varphi\|_\tau < \delta$ implies $|\tilde{J}^*| < \epsilon$. On the other hand, the optimal input $u$ will yield no larger cost than the selected candidate solution $u = 0$ and $J^* \geq 0$. Hence, for $\|\varphi\|_\tau < \delta$, we have $|J^*| \leq |\tilde{J}^*| < \epsilon$. Thus, $V$ is continuous at $x_t = 0$. \hfill $\square$
Lemma A.3 The optimal value functional $V(x_t) = J^*(x_t; t, \Sigma)$ of the open-loop finite-horizon optimal control problem (3.1), (3.2) satisfies in a neighborhood of the origin
\[ V(x_t) \geq \bar{V}(|x(t)|) \] (A.3)
in which $\bar{V}$ is a class $\mathcal{K}_\infty$ function.

Proof. Consider two regions around the origin given by
\[ \Omega_1 = \{x_t: \|x_t\|_{\tau} \leq \alpha\}, \quad \Omega_2 = \{x_t: \|x_t\|_{\tau} \leq 2\alpha\}. \]
Since $f$ is continuous and $\mathcal{M}$ is compact, $f$ is bounded by $|f| < M$ with some positive constant $M$ for all $x_t$ in $\Omega_1$ and $\Omega_2$. Now, let $x_t \in \Omega_1 \subset \Omega_2$. Then clearly $|x(t)| \leq \alpha$ and for $T_M = |x(t)|/2M$
\[ \frac{|x(t)|}{2} \leq |x(t')| \leq \frac{3|x(t)|}{2} \leq 2\alpha, \quad t' \in [t, t + T_M]. \]
From (3.3) and since $E \geq 0$, it directly follows that
\[ J(x_t, u; t, \Sigma) \geq \int_t^{\min(T_M, T)} \bar{F}(|x(t')|)dt' \geq \bar{F} \left( \frac{|x(t)|}{2} \right) \cdot \min \left\{ \frac{|x(t)|}{2M}, \Sigma \right\} =: \bar{V}(|x(t)|). \]

Lemma A.4 Suppose that Assumptions 3.1 and 3.2 are satisfied. For any sampling instant $t_i = i\Delta$ and $t^* \in [t_i, t_i + \Delta]$, the optimal value functional satisfies
\[ J^*(x_{i+}; t^*, \Sigma) \leq J^*(x_{i+}; t_i, \Sigma) - \int_{t_i}^{t^*} F(x(t'), u(t'))dt'. \]

Proof. Feasibility of the optimization problem is guaranteed by Lemma A.1. Let $x^*$ denote the state resulting from application of the optimal input $u^*$ starting from $x_{i+}$ at time $t_i$. The value of the objective functional at time $t^*$ is
\[ J^*(x_{i+}; t_i, \Sigma) = \int_{t_i}^{t^*} F(x^*(t'), u^*(t'))dt' + \int_{t^*}^{t_i+\Sigma} F(x^*(t'), u^*(t'))dt' + E(\tilde{x}_{t_i+\Sigma}). \] (A.4)
Let $\tilde{x}$ denote the state resulting from application of the feasible (suboptimal) input (A.1) starting at $x_{i+}$ at time $t^*$. The value of the objective functional at time $t^*$ for this suboptimal input reads
\[ \tilde{J}(x_{i+}; t^*, \Sigma) = \int_{t_i}^{t^*+\Sigma} F(\tilde{x}(t'), \tilde{u}(t'))dt' + E(\tilde{x}_{t_i+\Sigma}) \]
\[ = \int_{t_i}^{t^*+\Sigma} F(x^*(t'), u^*(t'))dt' + \int_{t^*+\Sigma}^{t_i+\Sigma} F(\tilde{x}(t'), \tilde{u}(t'))dt' + E(\tilde{x}_{t_i+\Sigma}). \] (A.5)
Combining (A.4), (A.5) and integrating (3.5) from $t_i + \Sigma$ to $t^* + \Sigma$ yields
\[ \tilde{J}(x_{i+}; t^*, \Sigma) \leq J^*(x_{i+}; t_i, \Sigma) - \int_{t_i}^{t^*} F(x^*(t'), u^*(t'))dt' \]
and because of suboptimality of $\tilde{J}$
\[ J^*(x_{i+}; t^*, \Sigma) \leq \tilde{J}(x_{i+}; t^*, \Sigma) \leq J^*(x_{i+}; t_i, \Sigma) - \int_{t_i}^{t^*} F(x^*(t'), u^*(t'))dt'. \]
Using these results, asymptotic stability as stated in Theorem 3.1 can now be proven.

**Proof of Theorem 3.1.** In the following, first stability of the closed loop is proven. Given $\varepsilon > 0$, assume without loss of generality that (A.3) in Lemma A.3 holds for all states in the neighborhood of the origin defined by $\|x_t\|_r < \varepsilon$ and define $\beta = \bar{V}(\varepsilon)$. Because of the continuity of $V$ at $x_t = 0$, it is possible to find a $\delta > 0$ such that $V(x_t) < \beta$ for all $\|x_t\|_r < \delta$. Due to Lemma (A.4), the optimal value functional $V(x_t) = J^*(x_t; t, \mathcal{T})$ satisfies along trajectories of the closed loop for all $t^* > t$

$$V(x_{t^*}) \leq V(x_t) - \int_t^{t^*} \hat{F}(|x(t')|)dt'.$$

(A.7)

Hence, it is non-increasing and therefore for all $t^* > t$

$$\|x_t\|_r < \delta \Rightarrow V(x_t) < \beta \Rightarrow V(x_{t^*}) < \beta \Rightarrow \|x_{t^*}\|_r < \varepsilon.$$

Thus, $x_t = 0$ is stable. In order to show asymptotic stability, use (A.7) iteratively to obtain

$$V(x_\infty) \leq V(x_t) - \int_t^{\infty} \hat{F}(|x(t')|)dt'.$$

(A.8)

Due to $V(x_\infty) \geq 0$ and $V(x_t)$ finite, the integral exists and is bounded. Because the closed loop is stable, $\|x_t\|_r$ is bounded for all time. With the input constraint set $\mathcal{W}$ compact and $f$ continuous, it follows that $f(x(t), x(t - \tau), u(t))$ is bounded for all $t > 0$. Hence, $x(t)$ is uniformly continuous which implies $\|x(t)\| \to 0$ as $t \to \infty$ according to Barbalat’s lemma (Khalil, 2002). □

**Appendix B**

**Proof of Lemmas 5.1 and 5.2.**

**Proof of Lemma 5.1.** The proof uses ideas given in De Souza & Li (1995). Since

$$x(t - \tau) = x(t) - \int_{-\tau}^{0} \dot{x}(t + \theta)d\theta$$

(B.1)

$$= x(t) - \int_{-\tau}^{0} \dot{f}(x(t + \theta), x(t - \tau + \theta), Kx(t + \theta)) + \Phi(x(t + \theta), Kx(t + \theta))d\theta$$

(B.2)

any solution of system (2.1) is also a solution of

$$\dot{\xi} = (A_k + A_T)\xi(t) + \Phi(\xi(t), K\xi(t))$$

$$- A_T \int_{-\tau}^{0} [A_k\xi(t + \theta) + A_T\xi(t - \tau + \theta) + \Phi(\xi(t) + \theta), K\xi(t) + \theta)]d\theta$$

(B.3)

$$\xi(\theta) = \psi(\theta), \forall \theta \in [-2\tau, 0]$$

(B.4)

in which the short hand $A_k = A + BK$ is used. Hence, if $\mathcal{Q}_\alpha$ is positively invariant for the latter system, then it is also positively invariant for the original system (2.1).

Define a Razumikhin function candidate $E_1$ as

$$E_1(\xi) = \xi^T(t)P\xi(t)$$

(B.5)

with the symmetric positive-definite matrix $P = A^{-1}$. The time derivative of $E_1$ along trajectories of (B.3), (B.4) is

$$\dot{E}_1(\xi) = \xi^T(t)\left[(A_k + A_T)^TP + P(A_k + A_T)\right]\xi(t) + 2\dot{\xi}^T(t)P\Phi(\xi(t), K\xi(t)) + \sum_{i=1}^{3} \eta_i(\xi, t)$$

(B.6)
in which
\[ \eta_1(\xi, t) = -2 \int_{-\tau}^{0} \xi^T(t) PA_\tau A_k \xi(t + \theta) d\theta, \]  
(B.7a)
\[ \eta_2(\xi, t) = -2 \int_{-\tau}^{0} \xi^T(t) PA_k^2 \xi(t - \tau + \theta) d\theta, \]  
(B.7b)
\[ \eta_3(\xi, t) = -2 \int_{-\tau}^{0} \xi^T(t) PA_\tau \Phi(\xi_{t+\theta}, K \xi(t + \theta)) d\theta. \]  
(B.7c)

For the symmetric matrices \( P_i = A_i^{-1} A_i A_i^{-1} > 0, i = 1, 2, 3 \), Inequality (5.4) yields \( P_i - P \leq 0, i = 1, 2 \). Furthermore, we know that for any \( v, w \in \mathbb{R}^n \) and for any symmetric positive-definite matrix \( P_i \in \mathbb{R}^{n \times n} \)
\[ -2v^T w \leq v^T P_i^{-1} v + w^T P_i w. \]  
(B.8)

Motivated by Razumikhin-type arguments assume that
\[ E_1(\xi(t + \theta)) \leq E_1(\xi(t)), \forall \theta \in [-2\tau, 0]. \]  
(B.9)

Thus, it follows from using (B.7–B.9)
\[ \eta_1(\xi, t) \leq \tau \xi^T(t) PA_\tau A_k P_i^{-1} A_k^T P_\tau \xi(t) + \tau \xi^T(t) P \xi(t), \]
\[ \eta_2(\xi, t) \leq \tau \xi^T(t) PA_k^2 P_i^{-1} (A_k^2)^T P \xi(t) + \tau \xi^T(t) P \xi(t), \]
\[ \eta_3(\xi, t) \leq \tau \xi^T(t) PA_\tau P_3 A_T \tau P \xi(t) + \int_{-\tau}^{0} \Phi(\xi_{t+\theta}, K \xi(t + \theta))^T P_3 \Phi(\xi_{t+\theta}, K \xi(t + \theta)) d\theta. \]

Substituting the result in (B.6) yields
\[ \dot{E}_1(\xi) < \xi^T(t)[\tau PA_\tau (A_k P_i^{-1} A_k^T + A_T P_\tau^{-1} A_k^T + P_\tau^{-1}) A_T \tau P + 2\tau P + (A_k + A_T)^T P + P(A_k + A_T)] \xi(t) \]
\[ + \int_{-\tau}^{0} \Phi(\xi_{t+\theta}, K \xi(t + \theta))^T P_3 \Phi(\xi_{t+\theta}, K \xi(t + \theta)) d\theta + 2 \xi^T(t) P \Phi(\xi_t, K \xi(t)). \]  
(B.10)

By using (B.9), we know that \( |\xi(t + \theta)| < v \xi(t) \) for all \( \theta \in [-2\tau, 0] \) with \( v^2 = \lambda_{\text{max}}(P)/\lambda_{\text{min}}(P) \). Using (4.4), we obtain
\[ 2 \xi^T(t) P \Phi(\xi_t, K \xi(t)) \leq 2 \|P\| \gamma (1 + v) \xi(t)^2 \]  
(B.11)
and
\[ \int_{-\tau}^{0} \Phi(\xi_{t+\theta}, K \xi(t + \theta))^T P_3 \Phi(\xi_{t+\theta}, K \xi(t + \theta)) d\theta \leq 4 \|P\| \gamma^2 v^2 \|P_3\| \xi(t)^2. \]  
(B.12)

Applying the Schur complement to (5.3), using the substitutions \( A = P^{-1}, A_i = P_i^{-1} P^{-1}, i = 1, 2, 3, \) and \( K = A_\tau A^{-1} \) and pre- and post-multiplying by \( P \) yields
\[ -W_1 := (A_k + A_T)^T P + P(A_k + A_T) + 2\tau P + \tau PA_\tau (A_k P_i^{-1} A_k^T + A_T P_\tau^{-1} A_T^\tau) \]
\[ + P_3^{-1} A_T^\tau P < 0. \]  
(B.13)
Now, choose $\alpha$ in (5.1) small enough such that for all states $x_t \in \Omega_\alpha$ the local control law satisfies the input constraints $u(t) = Kx(t) \in \mathcal{W}$ and the Property (4.4) holds with $\gamma$ small enough such that
\[
\Sigma_1(\gamma) + \Sigma_2(\gamma) < \lambda_{\min}(W_1)/2. \tag{B.14}
\]
Using (B.11), (B.12), (B.13) and (B.14) in (B.10), it can be ensured that $\dot{E}_1 < -\lambda_{\min}(W_1)|\xi(t)|^2$ whenever (B.9) holds. Note that (B.14) always holds for sufficiently small $\alpha$ because $P$ is positive definite and (4.4). Thus, by Razumikhin-type arguments, it follows that $\Omega_\alpha$ is positively invariant, see Hale & Lunel (1993).

\[\Box\]

**Proof of Lemma 5.2.** The derivative of $E$ in (5.5) along solutions of (2.1) when using the local control law $u = Kx(t)$ is
\[
\dot{E}(x_t) = x^T(t)[A_k^TP + PA_k]x(t) + 2x^T(t)PA_{\tau}x(t - \tau) + x^T(t)Sx(t) - x^T(t)Sx(t - \tau) + 2x^T(t)P\Phi(x_t, Kx(t)) \tag{B.15}
\]
Applying the Schur complement to the lower right block in (5.6), and pre- and post-multiplying by $\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ with $P = A^{-1}$, one obtains
\[
\begin{bmatrix} A_k^TP + PA_k + S + Q + K^T\!RK PA_{\tau} & A_k^TP \\ A_k^TP & -S \end{bmatrix} < -\varepsilon \mathcal{P}^2.
\]
Comparing this result to (B.15), it is clear that $\dot{E}(x_t) \leq -F(x(t), Kx(t))$ if
\[
2x(t)^T P\Phi(x_t, Kx(t)) \leq \varepsilon \lambda_{\min}^2(P) (|x(t)|^2 + |x(t - \tau)|^2). \tag{B.16}
\]
Arguments similar to the ones used in the proof of Lemma 5.1 yields
\[
|2x^T(t)P\Phi(x_t, Kx(t))| \leq 2|x(t)||P| \gamma |(x(t)^T, x(t - \tau)^T)| \leq 2|x(t)||P| \gamma (|x(t)| + |x(t - \tau)|) = \|P\| \gamma (2|x(t)|^2 + 2|x(t)||x(t - \tau)|) \leq \gamma \|P\| (3|x(t)|^2 + |x(t - \tau)|^2).
\]
Clearly, if $\alpha$ is chosen such that for all states $x_t \in \Omega_\alpha$ the local control law satisfies the input constraints $u(t) = Kx(t) \in \mathcal{W}$ and the Property (4.4) holds with $\gamma$ small enough such that
\[
\gamma \|P\| < 3\varepsilon \lambda_{\min}^2(P), \tag{B.17}
\]
then (B.16) holds and hence the assertion is true.

\[\Box\]

**Appendix C**

**Proof of Theorem 6.1**

Proof:

(a) Applying the Schur complement to the lower right block in (6.1), and pre- and post-multiplying by $\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$ with $P = A^{-1}$, one obtains
\[
\begin{bmatrix} A_k^TP + PA_k + S + Q + K^T\!RK PA_{\tau} & A_k^TP \\ A_k^TP & -S \end{bmatrix} < -\varepsilon \mathcal{P}^2. \tag{C.1}
\]
The derivative of the cost functional $E$ (6.2) along solutions of (2.1) with controller $u(t) = K x(t)$ is

\[
\dot{E}(x_t) = x^T(t)(A_k^T P + PA_k)x(t) + 2x^T(t)PA_t x(t - \tau) + x^T(t)Sx(t) - x^T(t - \tau)Sx(t - \tau) + 2x^T(t)P\Phi(x_t, K x(t)),
\]

in which $A_k = A + BK$. Comparing the results of (C.1) and (C.2), it is clear that $\dot{E}(x_t) \leq -F(x(t), K x(t))$ if

\[
2x^T(t)P\Phi(x_t, K x(t)) \leq \varepsilon \frac{\lambda_{\min}(P)}{2} \left| (x^T(t), x^T(t - \tau)) \right|^2.
\]

In order to show that this relation holds in the terminal region $\Omega$ defined by (6.3), note that property (4.4) is satisfied for all $\|x_t\|_\tau \leq \delta(\gamma)$ with $\gamma$ in (6.4). Therefore, the following holds:

\[
2x^T(t)P\Phi(x_t, K x(t)) \leq 2\lambda_{\max}(P) |x(t)| |\Phi(x_t, K x(t))| \\
\leq 2\lambda_{\max}(P) |x(t)| |(x^T(t), x^T(t - \tau))| \\
\leq 2\lambda_{\max}(P) |x(t)| |x^T(t), x^T(t - \tau))| \\
\leq \varepsilon \frac{\lambda_{\min}(P)}{2} \left| (x^T(t), x^T(t - \tau)) \right|^2.
\]

Hence, $\dot{E}(x_t) \leq -F(x(t), K x(t))$ for all $x_t$ for which $\|x_t\|_\tau \leq \delta(\gamma)$ and, hence, for all $x_t$ in the terminal region $\Omega$.

(b) In this part, the positive invariance of $\Omega$ is shown. The idea of this proof is based on the results of Melchor-Aguilar & Niculescu (2007). However, in this work, closed sets and controlled invariant regions are considered. Without loss of generality, assume that $x_{\mathcal{F}} \in \Omega$. For the sake of contradiction, assume that $\Omega$ is not positively invariant. Since $x(t)$ is a continuous function of time, there exists a $t_{\mathcal{F}} > t_{0}$ for which $x_{t_{\mathcal{F}}} \notin \Omega$ and $\|x_t\|_\tau < \frac{3\delta(\gamma)}{4}$ for all $t \leq t_{\mathcal{F}}$. Note that $\dot{E}(x_t) \leq 0$ for all $x_t$ with $\|x_t\|_\tau \leq \delta(\gamma)$ as shown in part (a) of this proof, hence

\[
E(x_{t_{\mathcal{F}}}) \leq E(x_{t_{0}}).
\]

Thus, at time $t_{\mathcal{F}}$, $\|x_{t_{\mathcal{F}}}\|_\tau > \delta(\gamma)/2$ because we assume $x_{t_{\mathcal{F}}} \notin \Omega$. It follows that there is a time $t_{2}$ with $t_{0} < t_{2} \leq t_{\mathcal{F}}$ for which

\[
|x(t_{2})| > \frac{\delta}{2},
\]

and due to $\dot{E} < 0$

\[
E(x_{t_{2}}) \leq E(x_{t_{0}}).
\]

On the other hand, closer inspection of the definition of the terminal cost functional in (6.2) gives $E(x_t) \geq \lambda_{\min}(P) |x(t)|^2$, and therefore

\[
E(x_{t_{2}}) \geq \lambda_{\min}(P) |x(t_{2})|^2 \geq \lambda_{\min}(P) \frac{\delta(\gamma)^2}{4}.
\]

Using this result and (C.5), it directly follows that

\[
E(x_{t_{0}}) > \lambda_{\min}(P) \frac{\delta(\gamma)^2}{4}.
\]
which contradicts the assumption that $x_{T_0} \notin \Omega$. Hence, the terminal region $\Omega$ is positively invariant. Furthermore, because of the invariance of $\Omega$ shown in (b) and since $\dot{E}(x_t) \leq -F(x(t), Kx(t))$, it directly follows that the control law $u(t) = Kx(t)$ locally asymptotically stabilizes the non-linear time-delay system (2.1).