Stability Analysis of Constrained Control Systems: An Alternative Approach.

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Abstract

The purpose of the present paper is to study stability of constrained nonlinear control systems. Usually, this is done by reducing the constrained control system to an unconstrained control system with respect to the constrained manifold. Different from these approaches, an alternative is proposed here which allows to establish stability without an explicit knowledge of the constrained manifold and the reduced unconstrained control system. The main result is a simple stability theorem. Despite it's simplicity, the theorem can be applied to a broad class of stability problems, for example, the stability analysis of unstructured nonlinear differential-algebraic equations of higher index and to LaSalle's invariance principle. Furthermore, the theorem allows a computationally efficient stability analysis of the class of constrained polynomial control systems using semidefinite programming and the sum of squares decomposition.

Key words: Constrained control systems, differential-algebraic systems, Lyapunov stability, polynomial control systems, minimum-phase analysis.

1 Introduction

In the study of systems and control theory, one often encounters problems which are governed by constrained dynamics subject to certain regions in the state space. Such constrained control systems (CCS) appear, for example, in modeling of electrical circuits and nonholonomic mechanical systems, or in

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the analysis of minimum-phase behavior and in optimal control. In the literature, one may find CCS under the synonyms descriptor systems, singular systems, differential-algebraic systems, semistate systems, or constrained dynamical systems. Although there are only few stability results available, for example (Müller 1998; Hanke and März 1996), more and more attention has been paid to CCS in recent years because of the increasing complexity in modern control systems. Basically, the stability problem of CCS is related to the problem of stability on manifolds, defined by the constraints in the control system description. For certain nonlinear CCS, a stability analysis can be done by using particular classes of Lyapunov functions candidates or by exploiting structural properties of certain nonlinear CCS (Bajic 1992). However, usually it is necessary to reduce the CCS to an unconstrained control system with respect to this constrained manifold (Rabier and Rheinboldt 1994; Clemente-Gallardo et al. 2001). Then, stability can be defined and analyzed in the same way as in the unconstrained case. The drawback of such a kind of stability analysis is, however, that one has to know or to compute the constrained manifold and the resulting unconstrained control system explicitly. This may be quite involved and usually only a local stability analysis is possible under certain regularity assumptions.

Different from these approaches, an alternative that avoids an explicit use of the constrained manifold and the reduced unconstrained control system is proposed in this paper. The main ingredients are so-called hidden constraints as well as some techniques from optimization theory. Combining these concepts, a simple yet general stability theorem in terms of a Lyapunov inequality for CCS is obtained. Despite it's simplicity and generality, the proposed approach has not yet been published before to the best of the authors' knowledge and no systematic study is found in the literature. The proposed approach is particularly attractive for computational purposes, since it allows an efficient stability analysis of the class of constrained polynomial control systems by semidefinite programming and the sum of squares decomposition. In the preliminary papers (Ebenbauer and Allgöwer 2004a, b), this approach was used to analyze stability of autonomous differential-algebraic systems and to give a new characterization of the minimum-phase behavior of nonlinear systems. In this paper, a more general framework is developed, which includes non-autonomous systems, with additional aspects of applications. For example, a variant of LaSalle's invariance principle is derived.

The structure of the paper is as follows: In Section 2, the problem formulation is given and the main stability theorem is derived. In Section 3, after some remarks on constrained polynomial control systems, several applications to the proposed approach, including a minimum-phase analysis and a stability analysis of switched systems, are given. Finally, the results of the paper are summarized in Section 4.

Notations. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive semidefinite, if $V(x) \geq 0$ for all x and positive definite, if $V(0) = 0$, $V(x) > 0$ for all nonzero x. Furthermore, V is called radially unbounded, if $V(x) \to \infty$ whenever $\|x\| \to \infty$. A continuously differentiable, positive definite, radially unbounded function V is called Lyapunov function candidate. If V is a differentiable function, then the row vector $\frac{\partial V}{\partial x}(x) = \nabla V(x)$ denotes the derivative of V with respect to x. $\mathcal{U}(\bar{x})$ denotes a neighborhood of a point $\bar{x} \in \mathbb{R}^n$, and for $\bar{x} = (\bar{x}_1, \bar{x}_2)$, $\mathcal{U}_{\bar{x}_1}(\bar{x})$ denotes a $\bar{x}_1$-neighborhood of a point $\bar{x} \in \mathbb{R}^n$, i.e., the unbounded set of all points $\{(x_1, x_2) \in \mathbb{R}^n \mid x_1 \in \mathcal{U}(\bar{x}_1)\}$. Furthermore, $\|x\|$ denotes the Euclidian norm of $x \in \mathbb{R}^n$ and $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$.

2 A Simple Stability Result for Constrained Control Systems

2.1 Problem Formuation

The CCS to be studied with respect to stability is of the form

$$f(x, \dot{x}, w) = 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $\dot{x}$ denotes the derivative of a trajectory $x = x(t)$ with respect of time. The input $w \in \mathbb{R}^m$ may be regarded as a control input or as disturbance. The function $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ is assumed to be sufficiently smooth with $f(0, 0, 0) = 0$.

The above formulation includes a broad class of control systems. In fact, one may even formulate switching behavior very easily, as discussed in Section 3. From solutions of (1), i.e., the trajectories $x = x(t)$, it is assumed that:

Assumption 1 *The trajectories $x = x(t)$ of (1) are absolutely continuous and sufficiently smooth almost everywhere and satisfy $f(x(t), \dot{x}(t), w(t)) = 0$ for almost all t on its right maximal interval of definition $[0, T_c)$ and for any admissible input $w = w(t)$.*

By “sufficiently smooth almost everywhere”, it is meant that the derivatives to be used are defined almost everywhere, e.g., trajectories which are smooth on intervals of positive length. Notice that such an assumption does not cover discontinuous solutions, which are well-known in CCS (Sastry and Desoer 1981). However, in principle, one could allow such jump behavior by assuming that there are, for example, only a finite number of jumps or by adding additional decreasing conditions in the statement to be presented. However, these conditions are avoided, since they cannot be easily verified on a computer and a much more involved mathematical analysis would be necessary. Furthermore, such a solution concept also implies that the input w must be admissible (consistent) with respect to the constrained dynamics. In particular, a control input $w = u(x,t)$ or a disturbance $w = d(x,t)$ has to be sufficiently smooth almost everywhere and is not allowed to violate the constraints of the dynamics. Otherwise, of course, the problem would not be well defined.

2.2 Basic Idea

The idea behind the proposed approach is simple and is illustrated in the following. The main ingredients are the incorporation of so-called hidden constraints or higher order constraints as well as some well-known techniques from optimization theory. Consider the following simple CCS as a motivating example:

$$\begin{aligned} \dot{x}_1 &= p_1(x_1,x_2) \\ \dot{x}_2 &= p_2(x_1,x_2,x_3) \\ 0 &= x_3 - q(x_1,x_2), \end{aligned} \tag{2}$$

with $x_i \in \mathbb{R}$, $p_i(0) = q(0) = 0$ and p_i, q appropriately defined. The constrained system (2) is visualized in Figure 1. There are two approaches to check stabil-

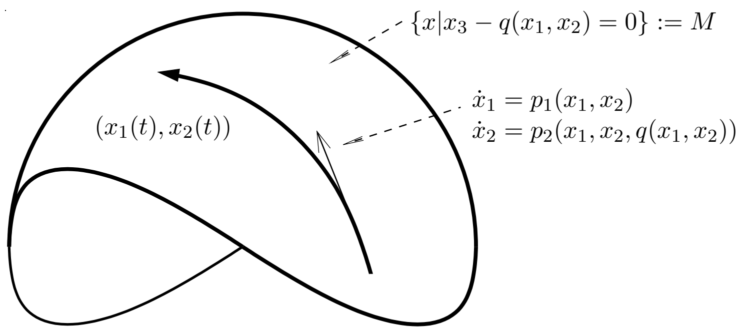

Fig. 1. Visualization of the constrained system (2).

ity. The standard ('explicit') approach is as follows: First, eliminate x_3 from the second equation, i.e., $\dot{x}_2 = p_2(x_1,x_2,q(x_1,x_2))$, and then check stability in the usual way by searching for a Lyapunov function $V = V(x_1,x_2)$ for $\dot{x}_1 = p(x_1,x_2)$, $\dot{x}_2 = p_2(x_1,x_2,q(x_1,x_2))$. Notice that this system represents the unconstrained system on the constrained manifold $x_3 - q(x_1,x_2) = 0$. Different to this, an alternative ('implicit') approach is the following: Instead of eliminating x_3, one may differentiate the third equation with respect to time, i.e., $0 = \dot{x}_3 - \frac{\partial q}{\partial x_1}(x_1,x_2)p_1(x_1,x_2) - \frac{\partial q}{\partial x_2}(x_1,x_2)p_2(x_1,x_2,x_3)$. Observe that this equation is satisfied for almost all $t \geq 0$, since $0 = x_3 - q(x_1,x_2)$ is satisfied for almost all $t \geq 0$. This procedure of generating additional equations by successive differentiation is well-known in the literature (Griepentrog 1992) and the obtained equations are sometimes called hidden constraints. Since $\dot{x}_3$ appears in the differentiated equation, one can adjoin this equation to the stability analysis, by searching now for a Lyapunov function $V = V(x_1, x_2, x_3)$ and a function $\rho = \rho(x_1, x_2, x_3)$ such that

$$\frac{\partial V}{\partial x_1}p_1 + \frac{\partial V}{\partial x_2}p_2 + \frac{\partial V}{\partial x_3}\left(\frac{\partial q}{\partial x_1}p_1 + \frac{\partial q}{\partial x_2}p_2\right) \leq \|x_3 - q\|^2 \rho \tag{3}$$

holds for some neighborhood around the origin $(x_1, x_2, x_3) = (0, 0, 0)$. The basic idea behind inequality (3) is to just check negative semidefiniteness of $\dot{V}$ with respect to the constrained set $\{x \mid x_3 - q(x_1, x_2) = 0\}$. Otherwise, if $x_3 - q(x_1, x_2) \neq 0$, one can always find a function ρ such that inequality above is satisfied, by just making $\rho(x_1, x_2, x_3)$ big enough. The idea of domination (penalization) is often used in optimization theory with constraints. For example, one can interpret ρ as a penalty or barrier function, or as Lagrange multiplier, or as a dual variable in the context of Lagrange-duality theory in optimization. In nonlinear systems and control theory, the use of similar ideas are quite old. Not exactly in the same way such an approach was used, at least implicitly, by Zubov to characterize the region of attraction via a partial differential equation (Hahn 1967). In the theory of linear matrix inequalities, one can find similar arguments for example in connection with the so-called S-procedure (Boyd et al. 1994). In (Papachristodoulou and Prajna 2002), a similar approach to study stability of constrained polynomial systems was proposed, however not for implicitly defined systems and without considering hidden constraints. Note that hidden constraints are crucial for the analysis of constrained dynamics and in particular of (singular) CCS. Coming back to the example, a general CCS is not always given in such a simple (semi-explicit) form as the example above. For instance, $\dot{x}_1, \dot{x}_2$ may not be obtained explicitly. Nevertheless, such situations can be handled by the same idea, as shown next.

2.3 Main Result

Define the following stacked vector of hidden constraints containing the first μ time derivatives of the vector field f:

$$F_\mu(\xi, \omega) = \begin{bmatrix} f(x, \dot{x}, w) \\ \frac{d}{dt} f(x, \dot{x}, w) \\ \vdots \\ \frac{d^\mu}{dt^\mu} f(x, \dot{x}, w) \end{bmatrix}, \tag{4}$$

with $\xi = (x, \dot{x}, ..., x^{(\mu+1)})$, $\omega = (w, \dot{w}, ..., w^{(\mu+1)})$, and $\frac{d}{dt}f(x, \dot{x}, w) = \frac{\partial}{\partial x}f(x, \dot{x})\dot{x} + \frac{\partial}{\partial \dot{x}}f(x, \dot{x})\ddot{x} + \frac{\partial}{\partial w}f(x, \dot{x}, w)\dot{w}$ and so on. Collecting now the ideas outlined above, one arrives at the following stability theorem for the CCS (1):

Theorem 1 *The equilibrium point $x = 0$ of the CCS (1) is stable for any admissible input $w = w(t)$, if there exist a Lyapunov function candidate $V : \mathbb{R}^n \to \mathbb{R}$, a function $\rho : \mathbb{R}^{(\mu+2)\cdot(n+m)} \to \bar{\mathbb{R}}$, and an integer number μ such that*

$$\nabla V(x)\dot{x} \leq \|F_\mu(\xi, \omega)\|^2 \rho(\xi, \omega) \tag{5}$$

is satisfied for some x-neighborhood $\mathcal{U}_x((\xi, \omega) = 0)$.

Proof. By Assumption 1, $f(x(t), \dot{x}(t), w(t)) = 0$ is satisfied almost identically on its right maximal interval of definition $[0, T_c)$. Hence $\frac{d}{dt} f(x(t), \dot{x}(t), w(t))=0$ and also for all higher order derivatives must hold: $\frac{d^i}{dt^i} f(x(t), \dot{x}(t), w(t))=0$ almost everywhere. Therefore, any trajectory $x = x(t)$ of the CCS (1) satisfies $F_\mu(\xi(t), \omega(t))= 0$ almost everywhere. By using the chain rule $\frac{d}{dt} f(x, \dot{x}, w) = \frac{\partial}{\partial x} f(x, \dot{x})\dot{x} + \frac{\partial}{\partial \dot{x}} f(x, \dot{x})\ddot{x} + \frac{\partial}{\partial w} f(x, \dot{x}, w)\dot{w}$ and so on, this implies $\nabla V(x(t))\dot{x}(t)$ ≤ 0 for almost all $t \geq 0$ in some x-neighborhood. It follows from this Lyapunov inequality and from the absolute continuity of the trajectories $x = x(t)$, that $V = V(x(t))$ is not increasing and hence the equilibrium point $x = 0$ of the CCS (1) is stable for any admissible input $w = w(t)$. □

Remark 1. The variables ξ, ω in the Lyapunov inequality (5) have to be considered as independent variables which are of purely algebraic nature, i.e., instead of $x^{(i)}$ one could write y_i.

Remark 2. Instead of differentiating the whole vector field f, it is sometimes convenient and sufficient to differentiate only some component functions f_i of the vector field $f = [f_1 \ldots f_n]^T$. For instance, one may differentiate these f_i's, where, in case of non-smooth inputs, the input does not appear.

Remark 3. In general, there is no universal rule how to choose μ and the f_i's, but in certain cases (cf. the minimum-phase analysis in Section 3) the choice of μ is clear. In (Pantelides 1988), an algorithm based on graph theory was proposed to find appropriate f_i's (see also (Campell et al. 1996)). However, notice that the larger μ can be chosen, the easier it is to satisfy the Lyapunov inequality (5). On the other hand, the analysis becomes more computationally expensive with increasing μ.

Remark 4. Observe also that Theorem 1 is a particular kind of robust stability theorem, since it guarantees stability for any admissible input w. It is desirable to use Theorem 1 in a more constructive way, namely, in stabilizing CCS. In principle, this can be done quite elegantly by incorporating the feedback $w = k(x)$ just as an additional (unknown) constraint. In the non-singular case $f(x, \dot{x}, w) = 0$, one would arrive at the following inequality

$$\nabla V(x)\dot{x} \leq \left\| \begin{matrix} f(x, \dot{x}, w) \\ w - k(x) \end{matrix} \right\|^2 \rho(x, \dot{x}, w), \tag{6}$$

where aside from V, ρ, one also has to search for a function k. Without going into details here, a computational problem appears, since the unknowns V, ρ, k do not enter linearly anymore in the Lyapunov inequality and hence a straightforward application of semidefinite programming and the sum of squares decomposition does not work anymore for the class of polynomial control systems.

Remark 5. Instead of taking the norm of F_μ in (5), one could also define the right side of (5) by an inner product between F_μ and a vector-valued ρ, i.e.,

$$\nabla V(x)\dot{x} \leq F_\mu(\xi, \omega)^T \rho(\xi, \omega) \tag{7}$$

which may be useful with regard to computational issues.

Remark 6. Notice that one has to satisfy Lyapunov inequality (5) in an x-neighborhood $\mathcal{U}_x((\xi, \omega) = 0)$ and not just in a neighborhood $\mathcal{U}((\xi, \omega) = 0)$. If one additionally assumes that $(\xi, \omega) = (\xi(t), \omega(t))$ becomes small whenever $x = x(t)$ becomes small, which is often the case in robustness considerations, then one can replace the x-neighborhood $\mathcal{U}_x((\xi, \omega) = 0)$ by neighborhood $\mathcal{U}((\xi, \omega) = 0)$ in Theorem 1.

Remark 7. For autonomous CCS, better known as differential-algebraic equations, i.e., $f(x, \dot{x}) = 0$, the notion of *index* plays a fundamental role in theoretical as well as numerical questions. Although no index theory is explicitly used here, there is a strong connection between the differentiation index and the stacked vector F_μ, which is outlined in the following. Under certain assumptions, a manifold is defined by the set of points x such that $F_\mu(x, \dot{x}, ..., x^{(\mu+1)}) = 0$ is solvable. This constrained manifold contains all trajectories $x = x(t)$ of the differential-algebraic equations $f(x, \dot{x}) = 0$. In (Griepentrog 1992; Griepentrog et al. 1992), this manifold was used to define the differentiation index. More precisely, the differentiation index μ is the smallest integer such that for every pair $(y_0, \eta_1, y_2, ..., y_{\mu+1})$, $(y_0, \eta_2, y_2, ..., y_{\mu+1})$ which satisfies $F_\mu(y_0, \eta_1, y_2, ..., y_{\mu+1}) = 0$ and $F_\mu(y_0, \eta_2, y_2, ..., y_{\mu+1}) = 0$ it follows that $\eta_1 = \eta_2$, i.e., $\dot{x}$ is unique. This allows a very intuitive geometric interpretation of the differentiation index and establishes the connection to the point of view of understanding a differential-algebraic equation as a implicitly defined vector field on an implicitly defined manifold. In particular, under certain assumptions, a unique vector field p on the constrained manifold can be determined such that $f(x(t), p(x(t))) = 0$ for $t \geq 0$ follows from $\dot{x} = p(x)$. (cf. (Griepentrog 1992; Rabier and Rheinboldt 1994)). In contrast to such an explicit characterization of this vector field p, one can consider the stacked vector F_μ as an implicit definition of this vector field p by setting F_μ to zero. The implicit stability analysis given by the Lyapunov inequality (5) then corresponds exactly to an explicit stability analysis of the vector field p on the constrained manifold. Notice also that Theorem 1 is applicable to differential-algebraic systems of higher-index. In case that the differentiation index of the differential-algebraic system is known, it is reasonable to choose μ to be equal to the differentiation index.

Remark 8. In the nonsingular case, i.e., for implicitly defined ODEs $f(x, \dot{x}) = 0$, where $\frac{\partial f}{\partial \dot{x}}$ has full rank (index 1 case), the Lyapunov inequality (5) simplifies to

$$\nabla V(x)\dot{x} \leq \|f(x, \dot{x})\|^2 \rho(x, \dot{x}), \tag{8}$$

which establishes stability for implicitly defined ODEs.

Remark 9. The assumptions of Theorem 1 are rather mild. Theorem 1 can be applied to CCS (1) when the solution is not unique or when μ is not equal to the differentiation index or even when the differentiation index is not well-defined. Furthermore, Theorem 1 can be also applied when the constrained dynamics is defined in a subset of the state space, and not on a smooth manifold. Moreover, one should notice that nontrivial or well-defined constrained dynamics need not always exist, e.g., $\dot{x} = x$, $0 = x^2$. Therefore, in such cases, one may speak instead of stability just about a convergence property established by the Lyapunov inequality (5) in Theorem 1.

Remark 10. Often it is of interest to establish asymptotic stability instead of stability. Then instead of (5), the strict inequality

$$\nabla V(x)\dot{x} < \|F_\mu(\xi, \omega)\|^2 \rho(\xi, \omega) \tag{9}$$

must be satisfied for all nonzero x in some x-neighborhood $\mathcal{U}_x((\xi, \omega) = 0)$.

Remark 11. For robustness and performance issues it may be useful to consider for example inequalities of the following type:

$$\nabla V(x)\dot{x} \leq -V(x) + \|F_\mu(\xi, \omega)\|^2 \rho(\xi, \omega) \tag{10}$$

instead of (5), (9), which, under additional assumptions on V (Hahn 1967), establishes exponential stability. In the same spirit, one can also investigate for example input-to-state stability or dissipativity theory of CCS.

Summarizing, Theorem 1 provides a fairly general framework for the stability analysis of constrained control systems using hidden constraints and domination (penalization) arguments from optimization theory. In the next section, several applications to Theorem 1 are given.

3 Applications

3.1 The Class of Polynomial Control Systems

In general, it is very difficult to search for a Lyapunov function V, a function ρ, and a number μ for practical problems. However, recently established methods based on semidefinite programming and the sum of squares decomposition allow to verify Lyapunov inequalities of the form (5) very efficiently in case f, V, and ρ are assumed to be polynomial (Parrilo 2000; Ebenbauer and Allgöwer 2004a). In the applications below, one may keep in mind to solve the obtained Lyapunov inequalities on a computer with the help of the sum of squares decomposition.

3.2 Analysis of the Minimum-Phase Property

Minimum-phase behavior is a fundamental and crucial notion in systems and control theory (Isidori 1994; Seron et al. 1999). Basically, the problem is to analyze asymptotic stability of a control system under the constraint that the output y is to be kept at zero. Consider the following control system affine in the input:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{11}$$

where f, g, h are assumed to be sufficiently smooth functions with $f(0) = 0, h(0) = 0$. Then, a minimum-phase analysis turns into the following CCS

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ 0 &= h(x), \end{aligned} \tag{12}$$

for which stability with respect to $x = 0$ has to be analyzed. Since $\dot{x}$ is given explicitly, it makes no sense to differentiate the first equation. However, it turns out that if the control system (11) has a well-defined normal form, then the control system (11) is minimum-phase if and only if

$$\nabla V(x)(f(x) + g(x)u) < \|F_r(x, u)\|^2 \rho(x, u) \tag{13}$$

is satisfied for all nonzero x in some x-neighborhood $\mathcal{U}_x((x, u) = 0)$, where r is the relative degree and F_r consists of all derivatives of the output y up to order r, i.e., $F_r(x, u) = [h(x), \dot{h}(x), ..., h^{(r)}(x)]^T$, $\dot{h}(x) = (\partial h(x)/\partial x)(f(x) + g(x)u)$ etc. (cf. (Ebenbauer and Allgöwer 2004b) for more details).

3.3 Switched Systems

Consider the following switched system:

$$\begin{aligned}\dot{x} &= f_0(x)(1-d) + f_1(x)d \\ 0 &= d(1-d).\end{aligned} \tag{14}$$

The switching behavior is defined by the second equation, i.e., d can only take the values 0 and 1. When arbitrary switching is allowed, the second equation is not differentiable. Applying Theorem 1 leads to the Lyapunov inequality

$$\nabla V(x)\left(f_0(x)(1-d) + f_1(x)d\right) \leq \|d(1-d)\|^2 \rho(x,d), \tag{15}$$

which in this case coincides with the problem of searching for a common Lyapunov function for the vector fields f_0 and f_1. Note that one can also handle inequality constraints like $d^2 \leq 1$, by just defining the equality constraint $d^2 + b^2 = 1$, where b is an additional variable [1].

3.4 LaSalle's Invariance Principle

LaSalle's invariance principle is an important tool to solve the problem of establishing asymptotic stability if only negative semidefiniteness of the derivative of the Lyapunov function along trajectories can be established. This kind of problem can be also formulated as a constrained problem. Consider the following system:

$$\dot{x} = f(x). \tag{16}$$

Let's assume that a Lyapunov function U is known such that $\dot{U}$ is negative semidefinite, i.e., $\nabla U(x) f(x) \leq 0$ for all x. The challenge is now to establish asymptotic stability. Using LaSalle's invariance principle, to be more precise, a result which goes back to Barbashin and Krasovskii (Hahn 1967), one has to check that no nontrivial trajectory $x = x(t)$ exists in the set where $\dot{U}$ vanishes. However, a more general statement is the following: If there is a trajectory $x = x(t)$ of (16) such that $\dot{U}(x(t)) = 0$ for all $t \geq 0$, then the trajectory has to converge to zero. This can be written as following constrained stability analysis problem for the following system:

$$\begin{aligned}\dot{x} &= f(x), \\ 0 &= \nabla U(x) f(x).\end{aligned} \tag{17}$$

Now, Lyapunov inequality (9) (instead of (5)) can be applied in the same fashion in the case of the minimum-phase analysis problem, where $\dot{U}$ plays the role of the output function h and, of course, r has to be replaced by μ.

[1] One may think about b as a dummy (free, unbounded) variable.

Lemma 1 *The stable system* (16) *with a smooth Lyapunov function U is asymptotically stable, if there exist a function ρ, a Lyapunov function candidate V, and an integer number μ such that*

$$\nabla V(x)f(x) < \|F_\mu(x)\|^2\rho(x) \tag{18}$$

is satisfied for all nonzero x, where $F_\mu(x) = [\dot{U}(x), \ddot{U}(x), ..., U^{(\mu)}(x)]^T$.

Proof. Follows from Theorem 1 and the fact that $\dot{x} = f(x)$ is stable. □

Notice that Assumption 1 is not satisfied since there are trajectories $x = x(t)$ which do not satisfy $\dot{U}(x(t)) = 0$, $t \geq 0$. However, these trajectories $x = x(t)$ are asymptotically stable. Moreover, asymptotic stability is established here with the help of two Lyapunov functions U and V, where the second Lyapunov function V investigates the trajectories, for which $\dot{U}(x(t))$ is identically zero. Furthermore, if the Lyapunov function U is only semidefinite, then one can also establish asymptotic stability by adding U in F_μ. Similar ideas to establish asymptotic stability from stability with the help of auxiliary functions can be also found in (Hahn 1967) (Matrosov's Theorem) and in (Loria et al. 2005). One can also consider Lyapunov inequality (9) as a kind of detectability criterion. This is underlined by the following interesting observation. Assume that a number μ is known, such that the norm of F_μ is positive definite (for all x), where $F_\mu(x) = [\dot{U}(x), \ddot{U}(x), ..., U^{(\mu)}(x)]$, then asymptotic stability can be established, since positive definiteness of $\|F_\mu(x)\|^2$ implies that there is no nontrivial trajectory $x = x(t)$ such that $\dot{U}(x) = 0$ for all $t \geq 0$. Observe that this means that system $\dot{x} = f(x)$ with the fictitious output $y = \dot{U}(x)$ has no zero dynamics.

4 Summary

Summarizing, in this paper a new approach for the stability analysis of constrained control systems was proposed which avoids an explicit decomposition of the CCS into a constrained manifold and the corresponding vector field defined on this manifold. The obtained Lyapunov inequality, despite it's simplicity, can be applied to a broad class of problems, as illustrated on several applications, including minimum-phase analysis of nonlinear systems and LaSalle's invariance principle. A further advantage of the proposed approach is that it allows a computer-aided stability analysis via semidefinite programming and the sum of squares decomposition for the class of constrained polynomial control systems. There are, of course, limitations in the proposed approach. First, in case of a non-polynomial system description, there are no efficient tools available to search for suitable functions V and ρ. Furthermore, also for the class of polynomial control systems, only small to mid-size problems can be tackled with current semidefinite programming algorithms, due to the fact that a lot of new variables enter into the analysis by differentiating the equations. This may become computationally demanding also for fairly small problems.

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