A Dynamical System That Computes Eigenvalues and Diagonalizes Matrices with a Real Spectrum

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***Abstract*— The present paper deals with the problem of diagonalizing matrices using a control system of the form $\dot{A} = [U, A]$, where $[U, A] = UA - AU$ and A, U are real matrices. It is shown that the feedback $U = [N, A + A^T] + \rho[A^T, A]$, N diagonal, $\rho > 0$ allows to solve the diagonalization problem under the assumption that the to be diagonalized matrix has real spectrum. Moreover, in the case of a complex spectrum, the feedback allows to check if a matrix is stable or to compute all eigenvalues of a matrix or roots of a polynomial.**

I. INTRODUCTION

The understanding of the computational power of dynamical systems and feedback is becoming increasingly important in many different research areas. Over the last decades, increasing attention has been paid, in areas like neuroscience, numerical mathematics, or cell biology, to systems which carry out computations in ways different from those based on digital logic (cf. e.g. [22], [16], [1], [3], [20], [13]).

In this paper, control systems of the form $\dot{A} = [U, A]$, where $[U, A] = UA - AU$ represents the Lie bracket (commutator) and U, A are real matrices, are utilized in order to solve a certain computational problem in a non-digital (analog) fashion, namely the problem of diagonalizing matrices. Systems of the form $\dot{A} = [U, A]$ are often called Lax systems. This class of systems has very interesting properties and appears in many areas of engineering and science, especially in mathematics, physics, and in control theory. Cf. for example [3] [2] and references therein.

The present work is closely related to Brockett's well-known work [7], see also [4],[6],[5],[10] (a more complete list of references can be found in [16]). In [7] the socalled double bracket flow $\dot{A} = [[N, A], A]$, where A is a symmetric matrix, N is diagonal, and $U = [N, A]$, has been introduced. This system has many remarkable properties. For example, one property is that the system allows to diagonalize real symmetric matrices in the following sense: if the initial condition is a symmetric matrix $A(0) = A_0$, then $A(t)$ converges to a diagonal matrix while preserving the spectrum of A_0. Over the last four decades, many systems have been derived in a way similar to that of [7] with many interesting properties. For example, well known are the Toda lattice, Oja's flow, or the QR flow. A collection of such systems and their properties can be found for example in the book by Helmke and Moore [16] as well as in the references therein. Related to the present results is also [17], where the non-symmetric case is considered. However, in contrast to the present paper, [17] does not guarantee existence and convergence of the solutions.

In this paper, the goal is to find a feedback $U = U(A)$ such that the control system $\dot{A} = [U, A]$ can be utilized not only to diagonalize symmetric matrices but also non-symmetric matrices with a real spectrum. The main result of this paper shows that such a feedback indeed exists. It has the form $U = [N, A + A^T] + \rho[A^T, A]$ and allows to diagonalize non-symmetric diagonalizable matrices, while preserving the spectrum. From the form of the feedback, it can be seen that the resulting new Lax system is a natural generalization of the double bracket flow. Moreover, for the case of a complex spectrum, it is shown that the derived systems can be used to the check if a matrix is stable or to compute, in an analog (continuous) fashion, eigenvalues of matrices or roots of polynomials.

The remainder of the paper is organized as follows: In Section II, the diagonalizing feedback and its properties (Theorem 1, 2, 3) is derived. In Section III, the main results are illustrated on various applications and a conclusion as well as an outlook is given in Section IV.

Notations. The transpose of a real $n \times n$ matrix $A \in \mathbb{R}^{n\times n}$ is denoted by A^T. The trace of a square matrix A is denoted by $trace(A)$ and the Frobenius norm of A is denoted by $\|A\|_F$, i.e., $\|A\|_F^2 = trace(A^T A)$. The conjugate of a complex $n \times n$ matrix $A \in \mathbb{C}^{n\times n}$ is denoted by $\overline{A}$ and the conjugate transposed by A^*. Moreover, $\Re\{A\}$ and $\Im\{A\}$ denotes the real, respectively, imaginary part of A and $[U, A] = UA - AU$ denotes the Lie bracket (commutator) with A, U square matrices.

II. MAIN RESULTS

Consider the control system

$$\dot{A} = [U, A], \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{R}^{n\times n}$ are arbitrary real $n \times n$ matrices. Then the isospectral property of (1) is a well-known fact, cf. e.g. [2], [11], [23]. For reasons of completeness, a proof is given under the additional assumption that the eigenvalues are pairwise distinct.

Lemma 1: Suppose that a given initial condition $A(0)$ of (1) has pairwise distinct eigenvalues, then (1) defines an isospectral flow.

Proof: Let $q(0), p(0)$ be the normalized ($q(0)^* p(0) = 1$) left and right eigenvector of an eigenvalue $\lambda(0)$ of $A(0)$. Then it is known [18] (Thm. 6.3.12) that if the eigenvalue

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has simple multiplicity, it changes according to:

$$\dot{\lambda}(0) = q(0)^* \dot{A}(0) p(0). \tag{2}$$

Therefore, one gets:

$$\begin{aligned} \dot{\lambda}(0) &= q(0)^*[U(0), A(0)]p(0) \\ &= q(0)^*(U(0)A(0) - A(0)U(0))p(0) \\ &= \lambda(0)(q(0)^*U(0)p(0) - q(0)^*U(0)p(0)) = 0. \end{aligned} \tag{3}$$

Hence the derivative of the eigenvalue $\lambda(t)$ at $t = 0$ is zero. This implies that the spectrum is preserved under the flow (1), because (3) is valid for each $t \in (0, T_{max})$, where T_{max} is the maximal interval of existence of (1). Notice, however, that the left- and right eigenvectors are not preserved, but the existence of the left- and right eigenvectors is guaranteed by the fact that $A(t)$ has the same *pairwise distinct* eigenvalues as $A(0)$. ■

The isospectral property of (1) is essential for the main results of this paper. The first result below shows that there exists a feedback U which allows to diagonalize non-symmetric matrices with real spectrum in the following (computational [22]) sense: if the initial condition (=input) $A(0) = A_0$ of (1) is a non-symmetric matrix with real distinct eigenvalues, then the flow (=process of computation) of (1) under a the feedback $U = [N, A + A^T] + \rho[A^T, A]$ converges to a diagonal matrix with the eigenvalues as diagonals (=output).

Theorem 1: Suppose that a given matrix $A_0 \in \mathbb{R}^{n\times n}$ has pairwise distinct real eigenvalues λ_i, i.e., A_0 can be written as $A_0 = T^{-1}\Lambda T$ with $\Lambda = diag(\lambda_1 \dots \lambda_n)$, $\lambda_i \in \mathbb{R}$, $T \in \mathbb{R}^{n\times n}$, T invertible. Then the solution $A = A(t)$ of (1) with the initial condition $A(0) = A_0$ and with the feedback

$$U = [N, A + A^T] + \rho[A^T, A] \tag{4}$$

converges to $\Lambda_\pi = diag(\lambda_{\pi(1)} \dots \lambda_{\pi(n)})$, i.e.,

$$\lim_{t\to\infty} A(t) = \Lambda_\pi, \tag{5}$$

where $(\pi(1), \dots, \pi(n))$ is a permutation of $(1, \dots, n)$, ρ is an arbitrary positive constant, and $N \in \mathbb{R}^{n\times n}$ is a real, diagonal matrix with pairwise distinct diagonals, i.e., $N = diag(n_1 \dots n_n)$, $n_i \neq n_j$ for $i \neq j$.

Proof: The idea of the proof goes, roughly speaking, as follows. "Normalization" step: First is it shown that the solution $A(t)$ defined by (1), (4), $A(0) = A_0$, satisfy $[A(t)^T, A(t)] \to 0$ for $t \to \infty$, which means $A(t)$ converges to the set of normal matrices, that is the set of matrices which satisfy $A^TA = AA^T$, or equivalently $A = \Theta^*\Lambda\Theta$, $\Theta^*\Theta = I$ [1]. Since the spectrum of $A(t)$ is preserved and real, $A(t)$ converges to the compact (bounded) set of symmetric matrices with a fixed spectrum Λ. "Diagonalization" step: Thus, $U(t) = [N, A(t)^T + A(t)] + \rho[A(t)^T, A(t)] \approx 2[N, A(t)]$ for $t \to \infty$. From which the diagonalization follows from [7].

[1]Since Λ is real, Θ is also real [18]. However, due to Theorem 2 the conjugate transposed is used instead of the transposed.

("Normalization") In a first step, it is shown that the derivative of the positive semidefinite function

$$\begin{aligned} V(A) =& \frac{1}{4} trace((A - A^T)^T(A - A^T)) \\ =& -\frac{1}{4} trace((A - A^T)^2) \end{aligned} \tag{6}$$

is monotonically decreasing along the flow (1) with the initial condition $A(0) = A_0$ and with the feedback (4) for all matrices A with $[A^T, A] \neq 0$. Differentiating (6) with respect to (1) and using the facts $trace(AB) = trace(BA)$ and $trace(A^T) = trace(A)$, one obtains:

$$\begin{aligned} \frac{d}{dt}V(A) =& -\frac{1}{2} trace((A - A^T)([U, A] - [A^T, U^T])) \\ =& -\frac{1}{2} trace((A^TA - AA^T)(U + U^T)) \\ =& -\frac{1}{2} trace([A^T, A](U + U^T)) \\ =& - trace([A^T, A]U) \\ =& - trace([A^T, A][N, A + A^T]) \\ & - \rho trace([A^T, A]^2). \end{aligned} \tag{7}$$

Now it is important to observe that

$$trace([A^T, A][N, A + A^T]) = 0. \tag{8}$$

This follows from the fact that the trace of a product between a symmetric matrix ($[A^T, A]$) and a skew-symmetric matrix ($[N, A + A^T]$) is zero. Thus with $trace([A^T, A]^2) = trace([A^T, A]^T[A^T, A]) > 0$ one gets

$$\frac{d}{dt}V(A) < 0 \ \forall [A^T, A] \neq 0. \tag{9}$$

Therefore, since V is bounded from below, the solution $A = A(t)$ of (1) under the action of feedback (4) converge into the set where the self-commutator vanishes, i.e.,

$$\lim_{t\to\infty} [A(t)^T, A(t)] = 0. \tag{10}$$

Since the spectrum of A is real, this implies that A converges to a symmetric matrix [18]. Notice that the limit (10) is well-defined, even though the Lyapunov function V is positive semidefinite only. To see this, assume that $\|A(t)\|_F^2 = trace(A(t)^TA(t)) \to \infty$ for $t \to T_{max}$, where T_{max} is the maximal interval of existence of $A(t)$. On the other hand, it is clear that V stays bounded for $t \to T_{max}$, i.e., $V(A(t)) \leq M$, $M \in \mathbb{R}$. This leads now to a contradiction since Lemma 1 ensures a constant spectrum, i.e., $\|A(t)\|_F^2 \to \infty$ implies $V(A(t)) \to \infty$ because:

$$\begin{aligned} V(A(t)) =& \frac{1}{4}\|A(t)^T - A(t)\|_F^2 \\ =& \frac{1}{4} trace(2A(t)^TA(t) - A(t)^2 - (A(t)^T)^2) \\ =& \frac{1}{2}\underbrace{\|A(t)\|_F^2}_{\to\infty} - \frac{1}{4}\underbrace{trace(\Lambda^2 + (\Lambda^*)^2)}_{constant} \to \infty. \end{aligned} \tag{11}$$

("Diagonalization") In a second step, it is shown that the function (compare with [7])

$$W(A) = \frac{1}{2} trace(N(A + A^T)) \tag{12}$$

and a solution $A(t)$ of (1), with the initial condition $A(0) = A_0$ under the action of the feedback (4), converges to a matrix A which satisfies $[N, A + A^T] = 0$. Differentiating (12) with respect to (1) and using the facts $trace(AB) = trace(BA)$, $trace(A^T) = trace(A)$, $[A, B] = -[B, A]$, $[A, B]^T = [B^T, A^T]$ one obtains[2]:

$$\begin{aligned}\frac{d}{dt}W(A) =&\frac{1}{2}trace(N[U, A]) + \frac{1}{2}trace(N[A^T, U^T])\\ =&\frac{1}{2}trace([A, N]U) + \frac{1}{2}trace([N, A^T]U^T)\\ =&\frac{1}{2}trace([A, N]([N, A + A^T] + \rho[A^T, A]))\\ +&\frac{1}{2}trace([N, A^T]([A + A^T, N] + \rho[A^T, A]))\\ =&\frac{1}{2}trace([A, N][N, A + A^T])\\ +&\frac{1}{2}trace([N, A^T][A + A^T, N])\\ +&\frac{\rho}{2}trace([A, N][A^T, A] + [N, A^T][A^T, A])\\ =&\frac{1}{2}trace([A + A^T, N][N, A + A^T])\\ +&\rho trace([A, N][A^T, A]).\end{aligned} \tag{13}$$

From (10) and (11) follows that any solution $A(t)$ exists for all $t > 0$ and it converges to the *bounded and invariant* set $\Omega = \{A : [A^T, A] = 0,\ \lambda(A) \in \{\lambda_1 \ldots \lambda_n\},\ V(A) \leq M\}$ defined by normal matrices with a certain (fixed) spectrum. Moreover, it is true that

$$\frac{d}{dt}W(A(t)) = -\frac{1}{2}trace([A(t) + A(t)^T, N]^2) \geq 0 \tag{14}$$

for solutions $A(t)$ in Ω. Thus, it follows from LaSalle's invariance principle [15] that solutions $A(t)$ starting in Ω converge to the set defined by $\frac{d}{dt}W(A(t)) = 0$, which implies

$$\lim_{t\to\infty}[N, A(t) + A(t)^T] = 0. \tag{15}$$

Therefore the set $\{A : [N, A + A^T] = 0\}$ is (conditionally) stable with respect to Ω. Using for example Theorem 3 (Remark 2, Remark 3) from [9] (see also [19]) with V given by (6), (9), it follows that (15) holds for any solution $A(t)$ not necessarily starting in Ω[3]. Thus, (10) to together with (15) implies that $A(t)$ converges, since $U \to 0$. Finally, it has to be shown that (15) implies that $A(t) \to \Lambda_\pi$. To see this, observe that [7], [16]

$$[N, A + A^T] = 0 \Leftrightarrow A + A^T = 2D \tag{16}$$

with D diagonal. In particular, $([N, A + A^T])_{ij} = (a_{ij} + a_{ji})(n_j - n_i) = 0$ if and only if $a_{ij} + a_{ji} = 0$, $i \neq j$. Thus, (15) implies that $A + A^T$ converges to a real diagonal matrix and since the spectrum of A is real, (10) implies that A converges to a symmetric matrix [18] (p.109, Problem 14). Hence by the isospectral property of (1), the desired result follows:

$$\lim_{t\to\infty} A(t) = \Lambda_\pi. \tag{17}$$

■

In the following, some remarks concerning Theorem 1 and its relation to [7] are discussed.

Remark 1: It is interesting to observe from the proof that the diagonalizing feedback (4) can be decomposed into two components

$$U = U_d + \rho U_n. \tag{18}$$

The second component $U_n = [A^T, A]$ plays the role of a "normalizer". This means, that this component forces $A(t)$ to become normal. The first component $U_d = [N, A + A^T]$ plays the role of a "diagonalizer". This means, that this component forces $A(t)$ to become diagonal. What is interesting (indeed the main idea) is the fact that the normalization component acts orthogonal to the diagonalization component. This can be seen, for example, in the proof in equation (8). The diagonalizer does not effect the normalization, i.e., the monotonicity of $\dot{V}$. In particular

$$U_d \in \Sigma^\perp, \quad U_n \in \Sigma, \quad U \in \Sigma^\perp \oplus \Sigma, \tag{19}$$

where U_d takes values in the set $\Sigma^\perp$ of skew-symmetric matrices and U_n takes values is the set Σ of symmetric (traceless) matrices. Hence

$$< U_d, U_n >= trace(U_d^T U_n) = 0. \tag{20}$$

This orthogonality relation, $U_n \perp U_d$, enables the simultaneous ("parallel") action of both, the normalization and the diagonalization process (see Fig. 1). Finally, the positive constant ρ plays simply the role of a relative gain between U_n and U_d. It is easy to see from the proof, that ρ can be even time-varying, i.e., $\rho = \rho(t)$ assuming that, for example, $\rho(t) \geq \rho_0 > 0$.

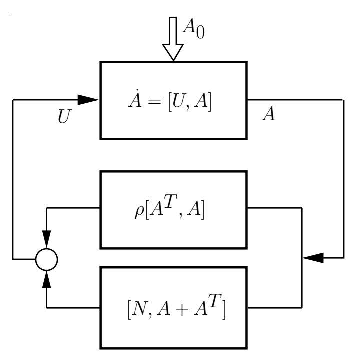

Fig. 1. Feedback loop for computation.

Remark 2: In contrast to the feedback $U = [N, A]$ used in [7], the feedback (4) is not obtained from a gradient type argument. It is, however, of interest to relate the current results

[2]One can simplify the calculation by using the fact $trace([A, N][N, A + A^T]) = trace([A^T, N][N, A + A^T])$.

[3]Essentially, (15) follows from LaSalle's invariance principle due to (9) and (14) and the global boundedness of $A(t)$, i.e., a monotonic and bounded sequence $(V(A(t)), W(A(t)))$ must converge (thus $\dot{V}(A(t)) \to 0, \dot{W}(A(t)) \to 0$).

to gradient flows and further investigations are needed. As mention in the introduction, if A is symmetric, then the self-commutator $[A^T, A]$ vanishes in (1) and (4), i.e., the "triple-A" term in $\dot{A} = [[N, A + A^T], A] + \rho[[A^T, A], A]$ disappears and one obtains the double bracket flow $\dot{A} = 2[[N, A], A]$ [7]. Consequently, the sorting behavior, i.e., the order of the eigenvalues along the diagonals of Λ_π is analogous to that of [7]. Confer [7] or [16] for more details. Finally, one can think of (1) and (4) as an (adjoint) flow on the general linear group $GL(n, \mathbb{R})$. Notice that the control U satisfies $trace(U) = 0$ and thus takes values in the Lie algebra (indeed in the Cartan decomposition (19) of the semisimple Lie algebra) $sl(n, \mathbb{R}) = \Sigma \oplus \Sigma^\perp$ of the special linear group $SL(n, \mathbb{R})$. In the literature, e.g. [8],[16],[2], the Lie group/algebra point of view is the usual way to look at systems like (1). Therefore, further investigations in this direction are interesting, i.e., the interpretation of the current results in the context of $sl(n, \mathbb{R}), su(n)_\mathbb{C} = \Sigma \oplus i\Sigma^\perp$.

What happens in the case of a complex spectrum? To see this, notice that the assumption made on A_0 in Theorem 1, i.e., a real spectrum, is not used in the proof of Theorem 1 until the very end. Therefore, the following statements can be made:

Theorem 2: Suppose that the eigenvalues of a given matrix $A_0 \in \mathbb{R}^{n\times n}$ have algebraic multiplicity one, where $A_0 = T^{-1}\Lambda T$ with $\Lambda = diag(\lambda_1 \ldots \lambda_n)$, $\lambda_i = \sigma_i \pm i\omega_i \in \mathbb{C}$, $T \in \mathbb{C}^{n\times n}$, T invertible. Then the symmetric component $A(t) + A(t)^T$ of a solution $A = A(t)$ of (1) with the initial condition $A(0) = A_0$ and with the feedback (4) converges to

$$\lim_{t\to\infty} A(t) + A(t)^T = 2\Re\{\Lambda_\pi\}. \tag{21}$$

Proof: From (16) follows that

$$A(\infty) = D + S, \tag{22}$$

where $A(\infty) = \lim_{t\to\infty} A(t)$ and S is a skew-symmetric matrix. Moreover, $A(\infty)$ is a normal matrix, i.e., $A(\infty) = \Theta^*\Lambda_\pi\Theta$, $\Theta^*\Theta = I$. Therefore,

$$\begin{aligned} 2D =& A(\infty) + A(\infty)^T = \Theta^*(\Lambda_\pi + \Lambda_\pi^*)\Theta \\ =& 2\Theta^*\Re\{\Lambda_\pi\}\Theta. \end{aligned} \tag{23}$$

Thus, the eigenvalues of D are $\Re\{\Lambda_\pi\}$. ∎

Moreover, under the assumption that the eigenvalues of A_0 must have *pairwise distinct real part*, i.e., an eigenvalue configuration like $\lambda_{1,2} = 3 \pm ia$, $\lambda_{3,4} = 3 \pm ib$, $a, b \in \mathbb{R}$, is not allowed, the structure of $A(\infty) = \lim_{t\to\infty} A(t)$ can be even more precisely predicted. To see this, observe first:

Lemma 2: Suppose $A = D + S$ is a normal matrix, where $D = diag(d_1 \ldots d_n)$ is diagonal and $S = (s_{ij})$ is skew-symmetric, then $([A, A^T])_{ij} = ([D, S])_{ij} = s_{ij}(d_j - d_i) = 0$, and consequently, if $d_j \neq d_i$, then s_{ij} is zero.

Proof: Since A is normal, $[A, A^T] = 0$. Thus,

$$[A, A^T] = [D + S, D - S] = 2[S, D] = 0. \tag{24}$$

Hence $([D, S])_{ij} = s_{ij}(d_j - d_i)$ must be zero. Consequently, for example, if $d_1 \neq d_i$, $i = 2...n$, then $s_{i1} = -s_{1i} = 0$. ∎

Since $A(\infty)$ is normal and $A(\infty) = D + S$, $D = \Re\{\Lambda_\pi\}$ (equation (22)), it is indeed possible, due to Lemma 2, to read off the eigenvalues of A_0 from $A(\infty)$.

Theorem 3: Suppose that the eigenvalues of a given matrix $A_0 \in \mathbb{R}^{n\times n}$ have *pairwise distinct real part* in the following sense: the eigenvalues of A_0 have algebraic multiplicity one and there exist no pairwise distinct indices i, j, k such that $\sigma_i = \sigma_j = \sigma_k$, where $A_0 = T^{-1}\Lambda T$ with $\Lambda = diag(\lambda_1 \ldots \lambda_n)$, $\lambda_i = \sigma_i \pm i\omega_i \in \mathbb{C}$, $T \in \mathbb{C}^{n\times n}$, T invertible. Then the solution $A = A(t)$ of (1) with the initial condition $A(0) = A_0$ and with the feedback (4) converges to the form

$$\lim_{t\to\infty} (A(t))_{ij} = \begin{cases} \sigma_{\pi(i)} & \text{if } i = j \\ \pm\omega_{\pi(i)} & \text{if } \sigma_{\pi(i)} = \sigma_{\pi(j)}, i \neq j \\ 0 & \text{else} \end{cases} \tag{25}$$

Proof: From equation (22) follows that $A(\infty) = D + S$ and $D = diag(d_1 \ldots d_n) = \Re\{\Lambda_\pi\} = diag(\sigma_{\pi(1)} \ldots \sigma_{\pi(n)})$. Moreover, $A(\infty)$ is normal. Thus, Lemma 2 can be applied. Notice that, due to pairwise distinct real parts, there exist no three pairwise distinct indices i, j, k such that $d_i = d_j = d_k$. Case 1 (complex eigenvalue): If $(A(\infty))_{ii} = \lim_{t\to\infty}(A(t))_{ii} = d_i = \sigma_{\pi(i)} = d_j$ for some i and $j (\neq i)$ in $\{1, ..., n\}$, then the *only* element in the ith column/row and jth row/column of S which does not vanish is s_{ij} ($s_{ji} = -s_{ij}$). In particular, for all k different from i and from j, Lemma 2 and pairwise distinct real parts imply $d_j - d_k \neq 0$ and $d_i - d_k \neq 0$ and thus $s_{kj} = 0$ and $s_{ki} = 0$ for all $k \neq i, j$. Case 2 (real eigenvalue): If for some i, $d_i = \sigma_{\pi(i)} \neq d_k$ for all $k \neq i$ then by Lemma 2, the ith column/row of S vanishes. By writing down $S = (s_{ij})$ and by taking into account that the eigenvalues of A_0 are identical with the eigenvalues of $A(\infty)$, it can be easily observed that the non-vanishing entries in S must be equal to $s_{ij} = \pm\omega_{\pi(i)}$. By relabeling the rows/columns of $A(\infty)$ such that $\sigma_{\pi(i)}, \sigma_{\pi(j)}$ are neighbors, in case $\sigma_{\pi(i)} = \sigma_{\pi(j)}$, $i \neq j$ (similarity transformation with permutation matrix), one can easily obtain the canonical form (2.5.9),(2.5.10) on p.105 [18]. Thus, the eigenvalues of A_0 can be read off from $A(\infty)$. ∎

Summarizing, Theorem 3 establishes the following structure of $A(\infty)$. If for example $d_k = \sigma_{\pi(k)}$ is simple (real eigenvalue $\lambda_{\pi(k)} = \sigma_{\pi(k)}$), then the kth column and row of $A(\infty)$ is zero except the diagonal element $(A(\infty))_{kk} = d_k = \sigma_{\pi(k)}$. If for example $d_i = \sigma_{\pi(i)} = \sigma_{\pi(j)} = d_j$, $i \neq j$, (complex eigenvalue $\lambda_{\pi(i)} = \sigma_{\pi(i)} \pm i\omega_{\pi(i)}$), then one obtains a structure of the form

$$A(\infty) = \begin{bmatrix} \ddots & & & & \\ & \sigma_{\pi(i)} & \cdots & \pm\omega_{\pi(i)} & \\ & \vdots & \ddots & \vdots & \\ & \mp\omega_{\pi(i)} & \cdots & \sigma_{\pi(j)} & \\ & & & & \ddots \end{bmatrix}. \tag{26}$$

III. APPLICATIONS

Simulation Example. The first example shows some numerical simulations for various gains ρ with

$$A_0 = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (27)$$

and $N = diag(1, 2, 3)$ (see Figure 2). From several simula-

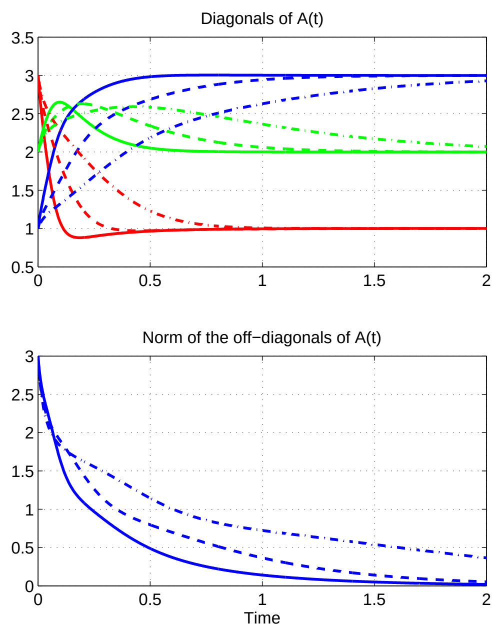

Fig. 2. Simulation results for (27) with $\rho = 0.5$ (dashdot), $\rho = 1$ (dashed), and $\rho = 2$ (solid).

tions it has been observed that the behavior of convergence is sensitive to the initial condition, i.e., the norm of A_0. For future research, time-varying gains $\rho = \rho(t)$, may be useful to improve convergence rates (performance) of the flow. Moreover, although Theorem 1 assumes that A_0 has to have pairwise distinct eigenvalues, numerical simulations show convergence also when the algebraic and geometric multiplicity is different, since an arbitrarily small perturbation of the eigenvalues, caused for example by numerical errors, may lead to algebraic multiplicity one. Thus, in numerical simulations, one can expect guaranteed convergence for any initial condition.

Application 1. From Theorem 2 follows that the diagonals of $A(\infty)$ correspond to the real part of the eigenvalues. Therefore, the system (1), (4) can be used to check if a matrix A_0 is stable (Hurwitz) by inspecting the diagonals of $A(\infty)$. Consider, for example, the matrix

$$A_0 = \begin{bmatrix} -1 & 2 & 1 & 1 \\ -2 & -3 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ -3 & -2 & -2 & 0 \end{bmatrix}. \qquad (28)$$

The eigenvalues of A_0 are 1, -2.0980, and $-0.9510 \pm 3.0175i$. Using $N = diag(1, 2, 3, 4)$ and $\rho = 1$, one obtains for $A(4)$ (see Fig. 3)

$$\begin{bmatrix} -2.0980 & 0.0002 & 0.0004 & 0.0000 \\ 0.0002 & -0.9513 & 3.0178 & 0.0000 \\ -0.0004 & -3.0177 & -0.9507 & -0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 1.0000 \end{bmatrix}. \qquad (29)$$

Since there is a positive diagonal element in $A(4)$, the matrix A_0 is not stable.

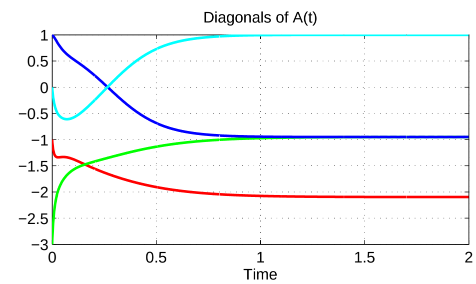

Fig. 3. Simulation results for (28).

Application 2. Finding roots of polynomials has a long history in science. Nowadays, any good scientific software package has a function like “root” to find roots of polynomials. However, before the advent of digital computers, the computation of roots of polynomials was not an easy task and needed considerable effort. The paper [14] entitled “machines for solving algebraic equations” gives a nice overview of the effort to build (electro)mechanical machines for finding roots of a polynomial, where the first machines were already build in the $18th$ century (see also [21]). With the results in this paper, it is possible to get another “machine” which computes (real and complex) roots of polynomials in an analog fashion. To see this, inspect the result above (matrix (29)). It can be seen that the imaginary parts of the eigenvalues can be read off as well. This is in general not the case, even though it happens (almost) always in simulations. However, it is true under the assumptions made in Theorem 3. Hence, the roots of polynomials can be immediately read off from $A(\infty)$, as demonstrated below. Consider for example the polynomial (see Fig. 4)

$$p(x) = x^4 - 3x^2 + x - 1. \qquad (30)$$

Forming the companion matrix

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 3 & 0 \end{bmatrix}, \qquad (31)$$

one obtains for $N = diag(1, 2, 3, 4)$ and $\rho = 1$

$$A(2) = \begin{bmatrix} -1.944 & 0.0006 & -0.0002 & 0.0000 \\ -0.0006 & 0.1442 & 0.5382 & -0.0005 \\ -0.0002 & -0.5382 & 0.1383 & 0.0014 \\ 0.0000 & -0.0005 & -0.0013 & 1.6615 \end{bmatrix}.$$

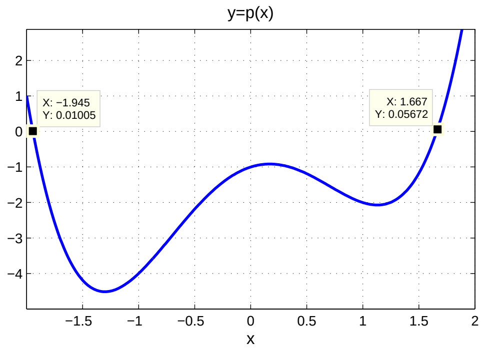

Fig. 4. Graph of (30).

From the matrix above, the structure of $A(\infty)$, as predicted by Theorem 3, can be clearly seen and the real roots as well as the complex roots can be read off from $A(2)$.
Another application is to check definiteness of a polynomial. Consider the polynomial

$$p(x) = x^4 - 2x^3 + x^2 + 3x + 3. \qquad (32)$$

Again, if A_0 is the corresponding companion matrix and $N = diag(1, 2, 3, 4)$ and $\rho = 1$, then one gets as solution

$$A(2) = \begin{bmatrix} -0.6146 & 0.5680 & -0.0001 & 0.0000 \\ -0.5676 & -0.6166 & 0.0003 & 0.0002 \\ 0.0001 & 0.0003 & 1.6156 & 1.2912 \\ -0.0000 & -0.0002 & -1.2912 & 1.6156 \end{bmatrix}.$$

Since there is no column/row with only a diagonal element in it, it follows that all eigenvalues have an imaginary part and since the polynomial p has even degree, it must be positive definite. Moreover, the complex roots can be read off because Lemma 2/Theorem 3 ($d_1 = d_2 \neq d_3 = d_4$) implies $s_{13} = s_{23} = s_{14} = s_{24} = 0$ ($s_{ij} = -s_{ji}$).

IV. CONCLUSION AND OUTLOOK

The main result of the present paper is a new Lax system which allows to diagonalize non-symmetric matrices and to compute their eigenvalues. Therefore, the present results extend the results for symmetric matrices [7] in a natural way. Moreover, the results can be helpful for designing new numerical algorithms [12]. The idea behind the new diagonalizing system is to define a feedback with a normalizing and a diagonalizing component such that the diagonalization process does not influence the normalization process. Some applications in the context of utilizing dynamical systems for computational purposes have been pointed out. For example, a stability test for matrices as well as roots computation for polynomials. Several points for future research has been discussed. In particular, the connections, interpretations, and generalizations of the present results in the context of Lie groups and Lie algebra.

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Extensions

Orthogonality is the key

Another diagonalizing flow is given by

$$\dot{A} = [\underbrace{A_l - (A_l)^T}_{skew}, A] + \rho[\underbrace{[A^T, A]}_{sym}, A].$$

A_l is strictly lower triangular part of A (Chu: $\dot{A} = [A_l - (A_l)^T, A]$).

Lie theoretic extension

The flow

$$\dot{A} = [[N, A + A^T], A] + \rho[[A^T, A], A]$$

can be rewritten as

$$\dot{X} = \rho[[N, Y], X] + [[Y, X], Y]$$

$$\dot{Y} = [[Y, X], X] + \rho[[N, Y], Y]$$

with $A = X + Y$, $X \in \Sigma^\perp$, $Y \in \Sigma$, due to the Cartan decomposition (semisimple Lie algebra), i.e. $[\Sigma, \Sigma] \subset \Sigma^\perp$, $[\Sigma^\perp, \Sigma^\perp] \subset \Sigma^\perp$ $[\Sigma, \Sigma^\perp] \subset \Sigma$. The flow converges to

$$[Y, X] = 0 \quad [Y, N] = 0.$$