Dissipation inequalities in systems theory: An introduction and recent results

Christian Ebenbauer, Tobias Raff, and Frank Allgöwer

Abstract. The aim of this article is to present an introduction to dissipation inequalities and to present some well known and some recent results in this area.

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1. Introduction

Lyapunov function techniques have received constantly high interest in applied mathematics and in particular in systems and control theory [8, 22, 38] over the last hundred years. The main reasons for this interest are simplicity, intuitive appeal, and universality of these techniques. Today, there is no doubt that Lyapunov functions techniques are the main tools to be used when one is faced with a stability or stabilization problem. In the analysis and design of control systems, however, there are usually other important requirements besides stability which have to be taken into account. Therefore, it is natural to ask the following question: Is it possible to generalize the ideas of Lyapunov function techniques in order to address for example robustness and performance issues in control systems? Such a generalization is indeed possible and has lead to the powerful concept of dissipativity and dissipation inequalities. Dissipativity has been introduced by Willems [35] and is motivated by the concept of passivity, a concept from electrical network theory which relates the stored energy in an electrical network with the supplied energy into the network. Alternatively, one can say that the basic idea behind dissipativity is to generalize the concept of Lyapunov functions techniques to systems with inputs and outputs. Over the past decades, dissipativity turned out to be an extremely useful concept in systems and control theory with plenty of applications in theory and practice.

The aim of this article is to give a brief introduction to dissipativity theory by discussing some well known and recent results in this area. In Section 2, the basic concept of dissipativity is presented. Some well known and recently established dissipation inequalities are presented in Section 3. Moreover, Section 4 presents some constructive and computational aspects of dissipativity theory. Finally, Section 5 concludes the article with a discussion and an outlook.

1.1. Notation. The notation used in this article is standard. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive semidefinite if $V(x) \geq 0$ for all x and positive definite if $V(0) = 0$, $V(x) > 0$ for all nonzero x. V is called proper (or radially unbounded) if $V(x) \to \infty$ whenever $\|x\| \to \infty$. A continuously differentiable, positive definite, radially unbounded function V is called a Lyapunov function candidate. For a scalar valued function $V : \mathbb{R}^n \to \mathbb{R}$, the row vector $\frac{\partial V}{\partial x}(x) = \nabla V(x) = [V_{x_1}(x), ..., V_{x_n}(x)]$ denotes the derivative of V with respect to x. A function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{K}$ if it is continuous, strictly monotonically increasing, and $\alpha(0) = 0$. $\mathcal{K}_\infty$ is the subset in the class of $\mathcal{K}$ functions that are unbounded. A function $\alpha : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in the first argument and if it is converging to zero whenever the second argument goes to infinity. $\|x\|$ denotes the Euclidean norm of a vector $x \in \mathbb{R}^n$. Furthermore, $P \geq 0$ ($P > 0$) indicates that a matrix P is symmetric and positive semidefinite (positive definite). Finally, $y(t) \equiv 0$, where $y : \mathbb{R} \to \mathbb{R}^q$, is used as a short form for $y(t) = 0 \; \forall t \geq t_0$.

2. Dissipation Inequalities

In systems and control theory one often encounters nonlinear control systems described, in the state space form, by means of a set of ordinary differential equations of the following type:

$$\begin{aligned} \dot{x} =& f(x) + G(x)u \\ y =& h(x), \end{aligned} \tag{1}$$

where $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^q$ is the output, and f, G, h are sufficiently smooth functions with $f(0) = 0, h(0) = 0$. The theory of dissipativity can be applied to a wider class of control systems, for example all results in this article can be extended to the system class $\dot{x} = f(x,u)$, $y = h(x,u)$. However, for the sake of exposition, only control systems of the form (1) are considered. The basic idea behind the definition of dissipativity is to establish a relation between the stored energy in a control system with the energy supplied into/dissipated by the system. In particular, the idea is to bound the increase of the stored energy by the supplied energy.

Definition 2.1. The control system (1) is said to be dissipative with respect to the supply rate $s : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$, if there exists a positive semidefinite storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that the (integral) dissipation inequality

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(x(t), u(t), y(t)) dt \tag{2}$$

is satisfied for all $t_0 \leq t_1$ and all solutions $x = x(t), u = u(t), y = y(t)$, $t \in [t_0, t_1]$, which satisfy (1).

It can be seen from Definition 2.1 that dissipativity involves three components: The first component is the positive semidefinite storage function V that can be interpreted as a generalized energy function. The second component is the supply rate s that can be interpreted as a generalized power supply and the third one is the dissipation inequality (2) that relates the storage function and the supply rate (see Figure 1). Note that the positive (semi)definiteness of the storage function V is not always necessary or desirable, but it is often needed in the context of stability. Furthermore, storage functions must not necessarily have a physical meaning (as in the case of Lyapunov functions), but there are canonical candidates for storage functions like energy or entropy.

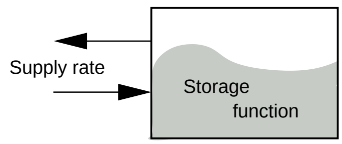


Figure 1. Illustration of supply rate and storage function.

If the storage function V is smooth then the integral dissipation inequality (2) can be rewritten as

$$\dot{V}(x(t)) \leq s(x(t), u(t), y(t)). \tag{3}$$

With no input and $s \equiv 0$, dissipation inequality (3) reduces to the Lyapunov inequality $\dot{V}(x(t)) \leq 0$. Thus, basically one can look at dissipation inequalities as generalized Lyapunov inequalities. But in contrast to Lyapunov inequalities, dissipation inequalities do not only summarize the internal quantities (states) of a control system but they also take into account the external quantities (inputs and outputs) and relate them to each other. Another particular appealing advantage of dissipation inequalities is the fact that the investigation of a possibly large number of differential equations, given by the control system description, boils down to a pointwise (or local, differential) dissipation inequality.

Definition 2.2. The control system (1) is said to be dissipative with respect to the supply rate $s : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$, if there exists a positive semidefinite storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that the following (differential) dissipation inequality is satisfied:

$$\nabla V(x)(f(x) + G(x)u) \leq s(x, u, h(x)) \; \forall x, u. \tag{4}$$

Notice that (3) is equivalent to (4) because $\dot{V}(x(t)) = \nabla V(x(t))\dot{x}(t)$. Therefore the solutions $x = x(t), u = u(t), y = y(t)$ are not needed, which is well known from Lyapunov's (asymptotic) stability condition $\dot{V} = \nabla V(x)f(x) \leq 0$ ($\dot{V} = \nabla V(x)f(x) \leq -\alpha(\|x\|)$, $\alpha \in \mathcal{K}_\infty$). Although the dissipation inequality (4) looks rather innocent, it is a powerful relation which allows to describe many important concepts and notions in systems and control theory using appropriate supply rates and storage functions. Due to the many applications, dissipation inequalities play nowadays a central role - most importantly because of their ability to characterize various aspects of optimality, robustness, and stability. More background about dissipativity theory can be found in many books in the control literature, e.g. [12, 16, 21, 23, 27, 36].

3. Characterization of System-theoretic Properties

The aim of this section is to define and to briefly discuss some important system-theoretic properties of control systems and to demonstrate how these rather different system properties can be considered from one single point of view when employing dissipation inequalities. In particular, the system properties passivity, L_2-gains, input-to-state stability, and minimum phase behavior are discussed in the subsequent sections. The structure of each subsection is as follows: The system property is first motivated from a system-theoretic point of view. Then, the property is defined and subsequently discussed and characterized by a dissipation inequality. Afterwards, some applications in systems theory are briefly pointed out. Finally, at the end of each subsection some references are given in order to provide more detailed information on these system properties.

3.1. Passivity. Passivity, originally a concept from electrical network theory, was first studied in control theory by Popov in the 1960's. The concept of passivity is motivated by the following consideration. Consider a RLC circuit, i.e. a circuit that consists of resistors, capacitors, and inductors as shown in Figure 2. The

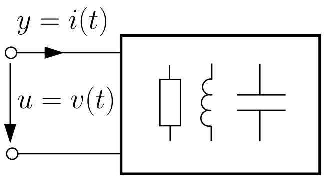


Figure 2. RLC circuit with power supply $p(t) = v(t)i(t)$.

power supplied into the system is given by $p(t) = v(t)i(t)$, i.e. the product of voltage and current. In electrical network theory, such a network (circuit) is said to be passive because it cannot supply more energy to its environment as energy was supplied to the network. This property can be expressed as

$$E(t_1) - E(t_0) \le \int_{t_0}^{t_1} v(t)i(t)dt, \quad t_0 \le t_1, \tag{5}$$

where $E(t)$ is the stored energy of the network at time t. In other words, (5) expresses the well known fact that RLC circuits cannot produce energy by their own. The generalization of (5) to arbitrary systems leads to the next definition.

Definition 3.1. The control system (1) with $p = q$ is said to be passive, if there exists a positive semidefinite storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that the following dissipation inequality is satisfied:

$$\nabla V(x)(f(x) + G(x)u) \leq u^T h(x) \ \forall x, u. \tag{6}$$

From this definition, one can see that the concept of dissipation inequalities, as defined in Definition 2.2, is a generalization of Definition 3.1. In the following, some properties of passive systems are pointed out. Firstly, because (6) must hold for *all* u's, one obtains the so-called nonlinear positive real lemma:

$$\begin{aligned} \nabla V(x)f(x) &\leq 0 \\ \nabla V(x)G(x) &= h^T(x). \end{aligned} \tag{7}$$

Secondly, it can be observed from (7) that passive systems are stable, assuming that V is positive definite. Moreover, if the control system (1) has a well-defined normal form [11], then it must be (weakly) minimum phase (see Section 3.4) and must have a vector relative degree of one, which means that the matrix $\frac{\partial h(x)}{\partial x}G(x)$ is invertible at $x = 0$ [4].

3.1.1. Applications. One very useful property of passive systems in systems theory is the fact that the parallel interconnection and the negative feedback interconnection of two passive systems is again a passive system (see Figure 3). This fact can be easily derived by defining the storage function V of the interconnected system as the sum of the two individual storage functions V_1, V_2, i.e. $V(x) = V_1(x) + V_2(x)$. For example, passivity of the negative feedback interconnection (see Figure 3, right) is shown as follows:

$$\dot{V} \leq u_1^T y_1 + u_2^T y_2 = (u - y_2)^T y_1 + y_1^T y_2 = u^T y. \tag{8}$$

This property is for example often used in large-scale network design of nonlinear interconnected systems and related topics, e.g. [1, 19]. Another important property of passivity is its relation to optimal control, cf. [27], p.95. Because of

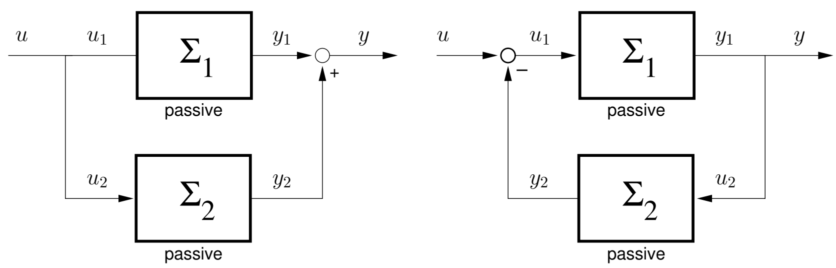


Figure 3. Parallel and negative feedback interconnection of passive systems are passive.

these and many other appealing properties and due to the fact that many systems in real-world are passive[1], passivity-based concepts have become popular approaches for systems analysis and feedback design. For example, in the recent work [32], passivity and dissipativity has been utilized for the analysis of global oscillations.

Summarizing, passivity is a useful system property and its generalization leads naturally to the concept of dissipativity. More background about passivity and its applications can be found for example in [4, 12, 14, 27, 33].

3.2. L_2-Gain. In many engineering applications it is of interest to know how a certain class of input signals is amplified or attenuated by a control system. For example, if the input signal u is an (undesired) disturbance signal, then one would like to know how much of this input disturbance can be sensed at the output (see Figure 4).

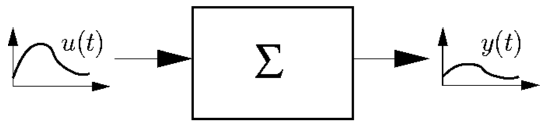


Figure 4. Input-output attenuation.

One way to characterize this property is by gains, i.e. by the quotient between output and input signals. In order to define this gain appropriately, one needs to specify the class of input signals and one needs to say what is meant by the size of a signal. In systems theory the class of Lebesgue integrable functions over the real interval $[0, \infty)$ defines such a reasonable class of signals. Especially, the class of square integrable functions $L_2[0, \infty)$, i.e. the class of functions for which

$$\|u\|_{L_2}^2 = \int_0^\infty \|u(t)\|^2 dt \tag{9}$$

is well-defined and finite, is of interest in systems theory. A measure for size of such a signal is usually its norm. The L_2-norm is of special interest, because signals with finite L_2-norm can be interpreted as finite energy signals, and thus this class of signals is physically meaningful. Using (9), the L_2-gain of a control system can be defined now as follows:

Definition 3.2. The control system (1) with $x(0) = 0$ is said to have an L_2-gain less or equal to γ if

$$\sup_{0<\|u\|_{L_2}<\infty} \frac{\|y\|_{L_2}}{\|u\|_{L_2}} \le \gamma. \tag{10}$$

[1]Eventually by choosing an appropriate fictitious output $y = \eta(x)$.

Thus, control systems with a finite L_2-gain γ satisfy for all L_2-input signals the input-output relation

$$\|y\|_{L_2} \leq \gamma \|u\|_{L_2}. \tag{11}$$

In the context of systems analysis and feedback design, one would like to characterize and compute the L_2-gain of a control system in a convenient way. In the following, it is shown that this is possible with the help of the dissipation inequality

$$\nabla V(x)(f(x) + G(x)u) \leq \gamma^2 \|u\|^2 - \|h(x)\|^2 \ \forall x, u, \tag{12}$$

under the assumption that the equilibrium point $x = 0$ of (1) is globally asymptotically stable and the storage function V is positive definite and proper. Integration of (12) leads to

$$V(x(t)) \leq V(x(0)) + \gamma^2 \int_0^t \|u(\tau)\|^2 d\tau - \int_0^t \|y(\tau)\|^2 d\tau. \tag{13}$$

From (13) follows that $x = x(t)$ is bounded and exists for all $t \geq 0$ if u is an L_2-signal. Moreover, since V with $V(x(0)) = 0$ is positive definite and proper, one obtains

$$\int_0^\infty \|y(\tau)\|^2 d\tau \leq \gamma^2 \int_0^\infty \|u(\tau)\|^2 d\tau. \tag{14}$$

Therefore, from (14) one gets (11) and the following theorem:

Theorem 3.3. *[33] The control system* (1) *has an L_2-gain less or equal to γ if there exists a positive definite and proper storage function V such that the following dissipation inequality is satisfied:*

$$\nabla V(x)(f(x) + G(x)u) \leq \gamma^2 \|u\|^2 - \|h(x)\|^2 \ \forall x, u. \tag{15}$$

Notice that Theorem 3.3 is a sufficient condition, however, for linear control systems it can be shown that it is also a necessary condition.

3.2.1. Applications. Similar to passivity, L_2-gains of systems are useful for the stability of interconnected systems. For example, it is often important in systems analysis to check if the negative feedback interconnection of two systems with finite L_2-gains is stable (see Figure 5). Using $V(x) = V_1(x) + \gamma_2^{-2} V_2(x)$ as a storage function, i.e. the weighted sum of the two individual storage functions, one can easily show (by calculating $\dot{V}$) that the feedback interconnection shown in Figure 5 is (asymptotically) stable if

$$\gamma_1 \gamma_2 < 1. \tag{16}$$

This result is known in the literature under the name small-gain theorem.

Summarizing, L_2-gain estimates can be obtained using dissipation inequalities. In particular they play a central role in the H_2 theory or H_∞ theory [26, 39] of linear control systems. More background about L_2-gains and applications can be found for example in [12, 33].

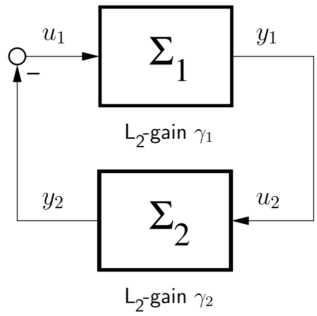


Figure 5. Stability of negative feedback loops.

3.3. Input-to-state Stability. From a practical point of view, asymptotic stability of an equilibrium point is often not very meaningful because small disturbances may cause an unstable behavior. Moreover, when dealing with systems with inputs, one would like to have a kind of stability notion which relates inputs and states. These considerations lead to bounded-input bounded-output/state stability. For example, it is often desirable to design control systems whose state is bounded in amplitude ($\|x(t)\| \leq N < \infty$ for all $t \geq 0$) if the input is bounded in amplitude ($\|u(t)\| \leq M < \infty$ for all $t \geq 0$). Control systems with this property do not have finite escape (blow-up) behavior for persistent (nonvanishing) input disturbances.

A simple sufficient condition to ensure this property is the existence of a positive definite and proper storage function V such that the following inequality holds:

$$\dot{V} = \nabla V(x)(f(x) + G(x)u) < 0 \quad \forall x : \|x\| \geq s \quad \forall u : \|u\| \leq r, \tag{17}$$

where $s = \rho(r)$ depends on the maximal input amplitude (see Figure 6). In other words, if $\|u\| \leq r$ then $\dot{V}$ is negative definite outside the ball $B = \{x : \|x\| \geq \rho(\|u\|)\}$, which (obviously) implies that $x(t) \to B$ as $t \to \infty$ [18], §25. The interesting fact about (17) is that the existence of such an inequality is also necessary for a bounded-input bounded-state behavior. Moreover, it can be shown that (17) is equivalent to a stability notion which is called input-to-state stability [20,29,31].

In the following, the relation between input-to-state stability and inequality (17) is explained in more detail, starting with the definition of input-to-state stability.

Definition 3.4. The control system (1) is said to be input-to-state stable, if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{0 \leq \tau \leq t} \|u(\tau)\|\right) \tag{18}$$

holds for all solutions $x = x(t)$ of (1).

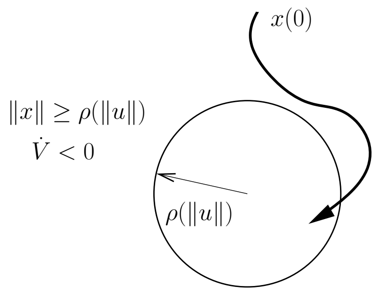


Figure 6. Bounded-input bounded-state idea.

In order to understand this definition, notice that the expression $\|x(t)\| \leq \beta(\|x(0)\|, t)$ is equivalent to global asymptotic stability [8]. Hence input-to-state stability implies global asymptotic stability ($u \equiv 0$). For $t \to \infty$ the influence of the initial state vanishes and the effect of the input amplitude remains and is measured in the γ term using the L_∞-norm. Therefore, for $t \to \infty$, one obtains the asymptotic gain

$$\|x(t)\| \leq \gamma\left(\|u\|_{L_\infty}\right). \tag{19}$$

In other words, for a bounded input (in the L_∞-norm) the state stays bounded. As pointed out before, a dissipation inequality like (17) is indeed equivalent to input-to-state stability, as stated in the next theorem.

Theorem 3.5. *[31] The control system* (1) *is input-to-state stable if and only if there exist a positive definite and proper storage function V and functions $\alpha \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ such that the following dissipation inequality is satisfied:*

$$\nabla V(x)(f(x) + G(x)u) \leq -\alpha(\|x\|) + \sigma(\|u\|) \ \forall x, u. \tag{20}$$

Notice that it can be shown that the dissipation inequality (20) is equivalent to (17)[2].

In the remainder of this subsection the sufficiency of Theorem (3.5) is discussed following the arguments of [30]. Due to simplicity of exposition the dissipation inequality

$$\nabla V(x)(f(x) + G(x)u) \leq -\alpha(V(x)) + \sigma(\|u\|) \ \forall x, u, \tag{21}$$

that is a slightly variated version of (20), is used to illustrate that (20) implies input-to-state stability. For any x, u, one can distinguish between two cases:

$$\sigma(\|u\|) \geq \frac{\alpha(V(x))}{2} \tag{22}$$

$$\sigma(\|u\|) < \frac{\alpha(V(x))}{2}. \tag{23}$$

[2]Simply by using the inequality $\max(a, b) \leq a + b \leq 2\max(a, b)$, $a \geq 0, b \geq 0$, and assuming that $\rho \in \mathcal{K}$.

In the first case, one obtains from (22) the estimate

$$V(x(t)) \leq \alpha^{-1}(2\sigma(\|u\|_{L_\infty})), \tag{24}$$

that is satisfied for any initial condition $x(0)$ and L_∞-signal $u = u(t)$ as long as (22) holds. In the second case, it follows from (20) that

$$\dot{V} = \nabla V(x)(f(x) + G(x)u) \leq -\frac{1}{2}\alpha(V(x)). \tag{25}$$

Moreover, as long as (23) holds (25) gives the estimate

$$V(x(t)) \leq \omega(V(x(0)), t). \tag{26}$$

Hereby, $\omega \in \mathcal{KL}$ is the solution of the initial value problem $\dot{\omega} = -\frac{1}{2}\alpha(\omega)$ with $\omega(0) = V(x(0))$. Hence, since $\dot{\omega} \geq \dot{V}$, $\omega(0) = V(x(0))$, one obtains (26). Thus, for $t \geq 0$, it follows from (24) and (26) that V is bounded from above by

$$V(x(t)) \leq \max\{\omega(V(x(0)), t), \alpha^{-1}(2\sigma(\|u\|_{L_\infty}))\}. \tag{27}$$

Since V is positive definite and proper, i.e. there exist functions $\bar{\alpha}, \underline{\alpha} \in \mathcal{K}_\infty$ such that $\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$, (27) can be rewritten as

$$\|x(t)\| \leq \max\{\underline{\alpha}^{-1}(\omega(\bar{\alpha}(\|x(0)\|), t)), \underline{\alpha}^{-1}(\alpha^{-1}(2\sigma(\|u\|_{L_\infty})))\}. \tag{28}$$

Finally, the inequality $\max(a, b) \leq a + b$ implies (18) with $\beta = \underline{\alpha}^{-1} \circ \omega \circ \bar{\alpha}$ and $\sigma = \underline{\alpha}^{-1} \circ \alpha^{-1} \circ 2\sigma$. Thus, the dissipation inequality (21) is a sufficient condition for input-to-state stability.

3.3.1. Applications. Input-to-state stability has found many applications in the systems and control literature. For example, analogous to the previous section, it is possible to formulate a small-gain theorem in terms of input-to-stability. In particular, the feedback interconnection

$$\begin{aligned} \Sigma_1 : \ \dot{x} &= f_1(x, z) \\ \Sigma_2 : \ \dot{z} &= f_2(x, z) \end{aligned} \tag{29}$$

is globally asymptotically stable if

$$\gamma_1(\gamma_2(s)) < s \tag{30}$$

holds for all $s > 0$, where γ_i $i = 1, 2$, are the gains of the input-to-state stable systems Σ_i, $i = 1, 2$. Another application of input-to-state stability is the stability analysis of cascaded feedback interconnections as shown in Figure 7. This feedback interconnection is globally asymptotically stable since the system Σ_2 is input-state-stable and thus no finite escape (blow-up) behavior can occur. Such a property is, for example, useful in observer-based output feedback design, where the system Σ_1 represents the dynamics of the observer error and Σ_2 represents the state feedback loop.

Summarizing, input-to-state stability, a notion in the spirit of bounded-input bounded-state behavior, can be characterized by the dissipation inequality (20). There are many other interesting aspects of this concept that have not been touched upon here and which can be found for example in [12, 30].

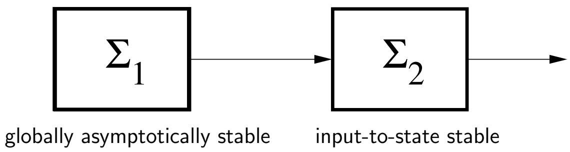


Figure 7. Globally asymptotically stable cascade.

3.4. Minimum Phase Property. When a control engineer encounters a control problem, one of the first steps in systems analysis might be to simulate the step response behavior. Doing so, it might happen that one can observe an inverse response behavior as shown in Figure 8. Roughly speaking, if an inverse response behavior can be observed, then it is more difficult to achieve a desired performance of the closed loop. Indeed, the distinction between easier and more difficult control problems is related to the inverse response behavior and can be made more precise using the notion of the minimum phase property. A control

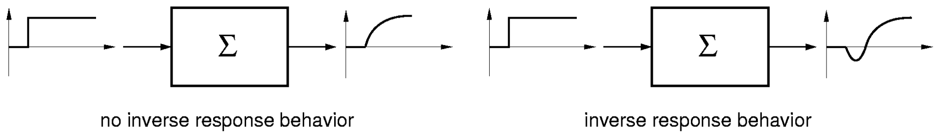


Figure 8. Inverse response behavior.

system is said to be minimum phase if it has an asymptotically stable zero output constrained dynamics (zero dynamics), which is obtained when the output of the system is kept identically equal to zero. For example in Section 3.1 it has been mentioned that passive systems must be (weakly) minimum phase. By inspecting (6) and by sending $y(t) = h(x(t))$ identically to zero, it follows that a passive system must be stable under the constraint dynamics $y(t) = h(x(t)) \equiv 0$, i.e. $y(t) = 0$, $t \geq 0$. Thus passive systems must be (weakly) minimum phase. For the special class of nonlinear control systems that are affine in the input and that posses a well-defined input-output normal form in the sense of [11], a rigorous definition of the minimum phase property can be given. The minimum phase property is then equivalent to the following definition.

Definition 3.6. The control system (1) is said to possess the minimum phase property with respect to the equilibrium point $x = 0$, if $x = 0$ is asymptotically stable under the constraint $y(t) \equiv 0$. The dynamics of (1) under this constraint is called zero dynamics.

In Figure 9, the zero dynamics is illustrated from a system-theoretic and from a geometric point of view. The system-theoretic interpretation is related to the output-zeroing problem, i.e. find an initial condition $x_z(0)$ and a control $u = u_z(t)$

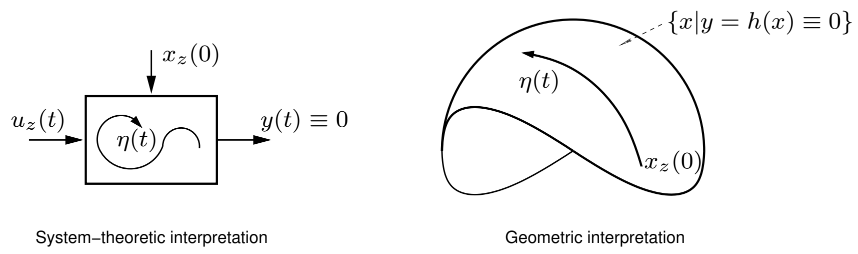


Figure 9. Various interpretations of the minimum phase property.

such that the output is identically zero (see Figure 9, left). In the geometric interpretation, the (global) minimum phase property is equivalent to the stability of the zero dynamics, i.e. the dynamics on the zero dynamics manifold defined by the constraint $y(t) \equiv 0$ (see Figure 9, right). In the context of dissipativity theory, the question arises now if it is possible to express the minimum phase property by means of a dissipation inequality. The next theorem gives an answer to this question. For simplicity, the single-input single-output is considered here, i.e. $p = q = 1$ in (1). Moreover, the notion of the relative degree r of an output must be defined beforehand. The relative degree measures how many integrators are between the input and the output. More precisely, the relative degree r of (1) for $q = p = 1$ is the minimal number that the output y must be differentiated until u appears in the expression, i.e. $y^{(j)}$ is not a function of u for $j < r$ but it is for $j = r$. For example, $\dot{y} = \dot{h}(x) = \nabla h(x)(f(x) + G(x)u)$ and if $\nabla h(0)G(0)$ is nonzero, then the relative degree at $x = 0$ is one. This means, that $\dot{y}$ can be fully controlled by u.

Theorem 3.7. *[7] Suppose that the relative degree r of (1) is well-defined at $x = 0$ (well-defined input-output normal form, [11],p.224). Then the control system (1) has the minimum phase property if and only if there exists a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and a smooth function $\rho : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{r+p}$ such that the dissipation inequality*

$$\nabla V(x)(f(x) + G(x)u) < H_r(x, u)^T \rho(x, u) \tag{31}$$

with $H_r(x, u)^T = \begin{bmatrix} h(x) & \dot{h}(x) & \ldots & h^{(r)}(x) \end{bmatrix}$ is satisfied for all u and all nonzero x in a neighborhood of $x = 0$.

Theorem 3.7 establishes a symmetric statement between the minimum phase property and the dissipation inequality (31). The dissipation inequality (31) is not difficult to understand. However, a few points have to be explained. Firstly, the role of the derivative array H_r is explained: The zero dynamics is the dynamics such that the output y is identically zero. This dynamics evolves on the zero dynamics manifold. Notice that the zero dynamics manifold is implicitly defined by $\|H_r(x,u)\| = 0$, since H_r is identically zero if $y(t) = 0$, $t \geq 0$. Imagine now ρ has the form $\rho(x,u) = H_r(x,u)\tilde{\rho}(x,u)$, which turns the inequality (31) into

$$\nabla V(x)(f(x) + G(x)u) < \|H_r(x,u)\|^2 \tilde{\rho}(x,u). \tag{32}$$

Therefore, stability on the manifold $\|H_r(x,u)\| = 0$ has to be studied, i.e. a Lyapunov function is needed subject to the constraint $\|H_r(x,u)\| = 0$. $\|H_r(x,u)\| > 0$ is not of interest. This situation is compactly expressed in inequality (31) ((32)), where ρ plays the role of a penalty function. Geometrically speaking, inequality (31) guarantees negative definiteness of the derivative of V only on a subset, namely on the set where $\|H_r(x,u)\| = 0$. For $\|H_r(x,u)\| > 0$, one can find a function ρ such that the left side is dominated by the right side of the dissipation inequality (31). These arguments are the underlying ideas to prove Theorem 3.7 [7]. Finally, it is worthwhile to point out the following fact. In systems and control theory it is important to distinguish between system properties which can be altered by feedback and system properties which cannot be altered by feedback. The minimum phase property cannot altered by feedback. One advantage of expressing the minimum phase property in terms of the dissipation inequality (31) is that this fact follows immediately. In particular, the inequality (31) must hold for *all* u's, thus no special u can turn a nonminimum phase system into a minimum phase system.

Summarizing, the minimum phase property, a notion which allows to describe fundamental performance limitations in control systems [2,28], can be characterized by the dissipation inequality (31). A robust variant of the minimum phase property can be found in [17]. Applications and more background about the minimum phase property can be found for example in [7, 11, 12].

3.5. Summary. As pointed out at the beginning of this section, dissipation inequalities allow to consider rather different system properties from a common point of view. To underline this important point, Table 1 summarizes the discussed system properties with their corresponding supply rates.

4. Computational Aspects

As in many areas of applied mathematics and engineering, the value of a concept for solving real-world problems often stands or falls with the availability of constructive or efficient computational techniques. In the case of dissipativity theory, one might think that the dissipation inequalities are more of theoretical interest since it is a well known fact that finding Lyapunov functions for general nonlinear systems is a difficult task. However, during the last decades, a lot of progress has been made in order to turn the concept of dissipation inequalities into a practically useful tool. In the following, three important system classes are pointed out, for which methods exist to construct or compute storage functions in a systematic way.

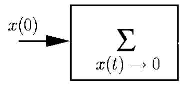
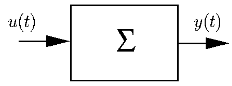
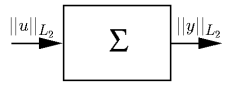
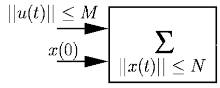
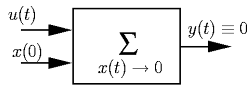
System Property	Supply Rate	Diagram
Asymptotic Stability	$-\alpha(\|\|x\|\|)$	
Passivity	$u^T y$	
L_2-Gain	$\gamma^2\|\|u\|\|^2 - \|\|y\|\|^2$	
Input-to-state Stability	$-\alpha(\|\|x\|\|) + \sigma(\|\|u\|\|)$	
Minimum Phase Property	$[y\ \dot{y}\ \dots\ y^{(r)}]^T \rho(x,u)$	

Table 1. System properties with their corresponding supply rates.

4.1. Linear Systems. The most well-studied class of control systems are linear time-invariant control systems given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{33}$$

where $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^q$ is the output, and A, B, C are matrices of appropriate dimension. For this class of systems, dissipativity theory can fully deploy its power because supply rates and storage functions can be computed efficiently. In particular, using linear matrix inequalities and semidefinite programming, dissipation inequalities can be solved efficiently. A semidefinite program, which can be seen as a generalization of a linear program, is a convex optimization problem and has the form

$$\begin{aligned} \text{minimize} \quad & c^T \xi \\ \text{subject to} \quad & F_0 + \sum_{i=1}^{k} \xi_i F_i \leq 0, \\ & D\xi = e, \end{aligned} \tag{34}$$

where $\xi \in \mathbb{R}^k$ is the unknown (decision) variable, $c \in \mathbb{R}^k$, $F_i = F_i^T \in \mathbb{R}^{m \times m}$, $D \in \mathbb{R}^{p \times k}$ are given matrices and $e \in \mathbb{R}^p$. In many situations in linear systems analysis and control design one can assume without lose of generality that the storage function is of the form

$$V(x) = x^T P x \tag{35}$$

with $P \geq 0$. Then, for example, the dissipation inequality (6) for passivity for system (33) turns into

$$2x^T P(Ax + Bu) \leq x^T C^T u \tag{36}$$

or equivalently into

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} PA + A^T P & PB - \frac{1}{2}C^T \\ B^T P - \frac{1}{2}C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0. \tag{37}$$

The symmetric matrix in the quadratic form (37) is a linear matrix inequality, since it is linear (affine) in the decision variable $P \geq 0$, and thus (37) can be rewritten in the form (34). Therefore, passivity of a linear time-invariant system can be checked using semidefinite programming by searching for a positive semidefinite matrix P such that (37) is satisfied. Many other dissipation inequalities, e.g. the dissipation inequalities for the L_2-gains or the minimum phase property, can be formulated as linear matrix inequalities in case the storage function is quadratic and the control system is linear. Nowadays, semidefinite programming and linear matrix inequality techniques are very successful in systems and control, and most of the basic problems in systems analysis and control design for linear time-invariant systems can be solved using semidefinite programming, cf. e.g. [3, 26].

4.2. Polynomial Systems. For linear time-invariant control systems, dissipation inequalities often reduce to quadratic forms. Recently, some of the ideas outlined in the previous section have been generalized to polynomial control systems. Polynomial control systems are control systems of the form (1) where f, G, h are polynomial functions of the state. Moreover, if the storage function and the supply rate are polynomial too, then dissipation inequalities of the form (4) reduce to algebraic (polynomial) inequalities. Thus, instead of asking whether a quadratic form is positive definite, the question arises whether a polynomial in several variables is positive definite or not? This question has a long history and goes back to Hilbert [10, 25]. From a computational point of view, one can show that the general problem of checking if a polynomial is positive definite is a hard problem. However, if a polynomial p can be written as

$$p(x) = \sum_i p_i^2(x), \tag{38}$$

i.e. as a sum of squared polynomials p_i, then the positivity of the polynomial p is obvious and, most importantly, it can be verified using semidefinite programming. This fact is summarized in the next theorem:

Theorem 4.1. *[5] A polynomial p of degree $2d$ has a sum of squares decomposition if and only if there exists a positive semidefinite matrix Q such that*

$$p(x) = m^T Q m, \tag{39}$$

where m is the vector of all monomials in $x_1, ..., x_n$ of degree less or equal to d, i.e. $m = [1, x_1, x_2, \dots, x_n, x_1 x_2, ..., x_n^d]$.

The proof of this theorem is not very difficult and is based on the Cholesky factorization $Q = P^T P$. Not all positive polynomials are sum of squares [25] but there are some special cases, like in the case of a single variable, where this is true. However, this representation theorem, sometimes called "Gram matrix" method, tells us that all sum of squares polynomials can be parameterized by the set (convex cone) of positive semidefinite matrices. Thus, Theorem 4.1 allows to check whether a polynomial is a sum of squares by applying semidefinite programming, i.e. by searching for a positive semidefinite matrix Q which satisfies (the linear constraints) $p(x) = m^T Q m$. For instant, searching for a Lyapunov function V for a polynomial system $\dot{x} = f(x)$ using sum of squares techniques means searching for a polynomial function V (of a certain total degree) such that the polynomials $V(x) - \|x\|^r$, $-\nabla V(x) f(x)$ are sum of squares polynomials. A term like $-\|x\|^r$, where r is an integer, is necessary in order to ensure positive definiteness.

In recent years, several work have been done in this area of research with the goal to extend the known analysis and design methods based on quadratic storage functions and linear systems to polynomial storage functions and polynomial systems. While some systems analysis techniques from linear systems can be transfered to polynomial systems, many questions remain open and are subject of current research effort. More details can be found in [6, 9, 24, 25].

4.3. Strict Feedback Systems. The methods presented in the previous two sections are based on numerical methods in order to solve dissipation inequalities. In this section, an analytical method is outlined which has been successfully applied in various fields of control, for example in the construction of storage functions in order to obtain input-to-state stable control systems. This method is called backstepping and it is applicable to strict feedback systems, i.e. to systems of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + G_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2)x_3 \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) + G_n(x_1, \ldots, x_n)u, \end{aligned} \tag{40}$$

where G_i are nonzero everywhere. Backstepping is based on the following result:

Theorem 4.2. *[12] Suppose that the control system* (1) *with* $p = 1$ *can be stabilized with a feedback* $u = k(x)$, $k(0) = 0$ *and that the corresponding Lyapunov function* V *of the closed loop is given. Then the augmented control system*

$$\begin{aligned} \dot{x} &= f(x) + G(x)z \\ \dot{z} &= u \end{aligned} \tag{41}$$

can be stabilized using the (control) Lyapunov function $V(x) + \frac{1}{2}(z - k(x))^2$ *and the feedback*

$$u = \nabla k(x)\left(f(x) + G(x)z\right) - \nabla V(x)G(x) - (z - k(x)). \tag{42}$$

The basic idea behind Theorem 4.2, which is often referred to as backstepping lemma, is to rewrite (41) as

$$\begin{aligned} \dot{x} &= f(x) + G(x)(k(x) + \zeta) \\ \dot{\zeta} &= v \end{aligned} \tag{43}$$

with $\zeta = z - k(x)$ and $u = \nabla k(x)(f(x) + G(x)z) + v$. From this it is easy to see that the time derivative of $V(x) + \frac{1}{2}\zeta^2$ along (41) can be rendered negative definite via the new control input v, using for example a feedback of the form (42). The idea behind backstepping is now to recursively apply the backstepping lemma to the control system (40). Doing so, firstly the subsystem

$$\dot{x}_1 = f_1(x_1) + G_1(x_1)u, \tag{44}$$

is stabilized and subsequently the augmented system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + G_1(x_1)x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + G_2(x_1, x_2)u \end{aligned} \tag{45}$$

via the backstepping lemma. Thus, using this recursive design idea, it is possible to systematically construct storage functions for control systems in strict feedback form which possesses a desired supply rate [12–14, 27].

5. Discussion and Outlook

The basic concept of dissipation inequalities has been explained in this article. Moreover, some well known and recent results in the characterization of system properties in terms of dissipation inequalities have been presented. In particular, the notion of passivity, L_2-gains, input-to-stability, and minimum phase behavior have been discussed in detail (see Table 1).

Since the purpose of this article has been to give a brief introduction to dissipation inequalities with some recent results, many topics and areas where dissipativity plays an important role have not been discussed here. For example, in the area of port-Hamiltonian systems [34], ideas related to dissipativity are also employed. Moreover, not covered in the current article is the relation between dissipativity and optimal control, i.e. the relation between optimal value functions and storage functions. The aspect of control design has not been discussed either, i.e. finding a control law $u = k(x) + v$ such that the obtained control system has a certain supply rate with respect to the new input v. Another point not discussed in detail are fundamental limitations of the concept of dissipativity. For example, which system properties cannot be conveniently characterized by dissipation inequalities? A complete answer to this question is probably not possible. However, one fundamental limitation of dissipativity is the fact that it is, at least in one of its most useful forms (4), a first order concept (first derivative of V). While this is almost always satisfactory in linear control, it has its limitations in nonlinear control where second and higher order effects are often of central importance, like in controllability and reachability analysis. Furthermore, the idea behind dissipation inequalities is intrinsically related to Lyapunov function techniques. Thus, these system properties which can be conveniently characterized by dissipation inequalities seem to have always a stability-like flavor. On the other hand, Lyapunov function techniques are not only useful for stability-related questions, as this is well known from the literature, cf. e.g. [15, 37]. These points might be a source for future research in the area of dissipation inequalities. Additionally, one of the main challenges for the future is to improve computational techniques in order to make dissipation inequalities more appealing for practical control-engineering purposes. Looking ahead, dissipativity will definitively continue to play an important role in many areas of systems theory and control, simply because new system properties and notions will automatically trigger the question: Is it possible to express these system properties in the language of dissipativity?

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Christian Ebenbauer is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, USA. Christian Ebenbauer was financially supported by the FWF Austria.

E-mail: ebenbauer@mit.edu

Tobias Raff and Frank Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Germany.

E-mail: {raff,allgower}@ist.uni-stuttgart.de