# A Dissipation Inequality for the Minimum Phase Property of Nonlinear Control Systems and Performance Limitations

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***Abstract*— The minimum phase property is an important notion in systems and control theory. In this paper, a characterization of the minimum phase property of nonlinear control systems in terms of a dissipation inequality is derived. It is shown that this dissipation inequality is equivalent to the classical definition of the minimum phase property in the sense of Byrnes and Isidori, if the control system is affine in the input and the so-called input-output normal form exists. Furthermore, it is shown that in case of linear control systems the derived dissipation inequality allows to establish a connection to Bode's T-integral. Thus the dissipation inequality can be utilized to quantify fundamental performance limitations in feedback design.**

***Index Terms*— Minimum Phase Property, Dissipation Inequalities, Performance Limitations, Bode's T-integral**

## I. Introduction

BODE introduced the notion of minimum phase property in his seminal paper [6] more than 60 years ago. Today, the minimum phase property plays an important role in systems analysis and control design [15], [17], [16], [30]. For example, the notion of the minimum phase property can be used to describe fundamental performance limitations in feedback design (e.g., [6], [23], [18], [11], [10], [21], [26], [27], [2], [29], [1]) and thus allows, roughly speaking, to distinguish between easy and difficult control problems. For linear time-invariant single-input-single-output systems, the minimum phase property is characterized for example by all zeros of the transfer function being in the open left half plane. The notion of zeros was generalized by Byrnes and Isidori (cf. e.g. [15]) to nonlinear control systems. For nonlinear control systems, loosely speaking, a system is said to be minimum phase if it has asymptotically stable zero output constrained dynamics (zero dynamics), which is obtained when the output of the system is kept identically equal to zero. For the special class of nonlinear control systems that are affine in the input and that posses a well-defined input-output normal form in the sense of [15], a rigorous definition of the minimum phase property can be given. In the following, this situation is referred to as the minimum phase property in the sense of Byrnes-Isidori. The minimum phase property is then equivalent to the situation that an equilibrium point, let's say $x = 0$, is asymptotically stable under the constraint that the output $y(t) = 0$, $t \geq 0$. In general, however, a precise definition of the minimum phase property for general nonlinear control systems is not an easy task. The reason for this is that the zero dynamics may not be well-defined, and even if this were the case, it makes no sense to speak about stability without saying something about equilibrium points (or sets). Beside this, it may be difficult to check if a control system is minimum phase or not. In the literature (cf. e.g. [15]), there exist at least two strategies for a minimum phase analysis: The first one makes use of a transformation of the control system into the input-output normal form, if the normal form exists. The second one is based on simply setting $y(t), \dot{y}(t), \ldots$ to zero, i.e, by setting the output and its higher order Lie-derivatives to zero and by calculating the remaining dynamics, which is equivalent to the zero dynamics. The second strategy is more general, since it also works when a transformation into the input-output normal form does not exist.

In this paper, a new third possibility is given to characterize the minimum phase property, namely in terms of a dissipation inequality. It is shown that the definition of the minimum phase property in the sense of Byrnes-Isidori for affine control systems with a well-defined input-output normal form is equivalent to the fact that a certain dissipation inequality is satisfied. Hence the minimum phase property, which has its origin in the frequency domain world and in geometric control, is expressed in terms of a Lyapunov-based language in this new approach. Moreover, the dissipation inequality can be easily applied to general nonlinear control systems that are not necessarily affine in the input. Furthermore, it is shown that for single-input single-output linear, time-invariant control systems the dissipation inequality allows to establish a connection to Bode integrals and to cheap control [27]. In particular, an alternative viewpoint on characterizing performance limitations due to the nonminimum phase property is discussed in this paper. The idea is to search for a new output such that a given nonminimum phase control system becomes minimum phase and such that the new minimum phase output is the closest one to the true nonminimum phase output in the $L_2$-sense. This discussion is carried out using duality theory from convex optimization and by utilizing the established dissipation inequalities for the minimum phase property. Thus, a new viewpoint is proposed which enables to quantify fundamental performance limitations in feedback design. In addition to the preliminary work [8], an additional result on the smoothness

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of a certain function involved in the derived dissipation inequality is established. This result plays an important role when using the dissipation inequality in a constructive or in a computational way. Furthermore, to the authors best knowledge, the proposed approach which establishes a connection to performance limitations using duality theory and the obtained dissipation inequality for the minimum phase property is new.

The only known results where the minimum phase property is expressed in terms of a Lyapunov-based language, i.e., in terms of a dissipation inequality are [19] and [8]. In [19] another alternative (stronger) notion of minimum phase property is given, based on output-input-stability which is in the spirit of Sontag's "input-to-state stability" philosophy. In particular a dissipation inequality is used in [19], which is a sufficient condition for the minimum phase property. The dissipation inequality there is, however, not a necessary condition, since the notion used there is motivated by introducing additional robustness in the minimum phase property. Therefore, the dissipation inequality derived there does not fully coincide with the well-established notion of minimum phase property in the sense of Byrnes-Isidori. In the preliminary work [8], a dissipation inequality is derived which is necessary and sufficient for the minimum phase property and which is slightly different from the dissipation inequality derived below. However, in contrast to [8], the results established in Section III, in particular Theorem 2 allow an additional smoothness statement of a function that appears in the dissipativity characterization of the minimum phase property.

The structure of the paper is as follows: In Section II, results from the literature are revisited and the class of control systems to be considered, the input-output normal form, and the definition of the minimum phase property in the sense of Byrnes-Isidori is given. In Section III, the dissipation inequality which characterizes the minimum phase property is derived. In Section IV, the connection to a Bode integral and to cheap control is established for single-input single-output linear, time-invariant systems based on a new dual viewpoint. Some numerical examples demonstrate the results of this paper in Section V. Finally, Section VI concludes with a discussion and summary.

## II. Preliminaries

The class of control systems studied in this paper is of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $f:\mathbb{R}^n \to \mathbb{R}^n$, $G:\mathbb{R}^n \to \mathbb{R}^{n\times p}$, $h:\mathbb{R}^n \to \mathbb{R}^p$ and $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, and $y \in \mathbb{R}^p$ is the output.

The main assumption on the control system (1) is that an input-output normal form exists:

*Assumption 1:* The functions $f, G, h$ in (1) are assumed to be sufficiently smooth with $f(0) = 0$, $h(0) = 0$ and furthermore, it is assumed that there exists a local change of coordinates $[\xi,\eta]^T = \Phi(x)$ with $\Phi(0) = 0$, $\Phi$ sufficiently smooth, such that the control system (1) with the same number of inputs and outputs can be represented in input-output normal form ([15], p.224):

$$\begin{aligned} \dot{\xi}_1^i &= \xi_2^i \\ &\vdots \\ \dot{\xi}_{r_i-1}^i &= \xi_{r_i}^i \\ \dot{\xi}_{r_i}^i &= b_i(\xi,\eta) + \sum_{j=1}^{p} a_{ij}(\xi,\eta)u_j \\ \dot{\eta} &= q(\xi,\eta) + P(\xi,\eta)u \\ y_i &= \xi_1^i, \end{aligned} \tag{2}$$

where $\xi = [\xi_1^1 \ldots \xi_{r_1}^1, \xi_1^2 \ldots]^T$, $i = 1 \ldots p$. Moreover, it is assumed that $q(0,\eta) - P(0,\eta)A(0,\eta)^{-1}b(0,\eta)$ is sufficiently smooth, with the square invertible (decoupling) matrix $A(\xi,\eta) = (a_{ij}(\xi,\eta))$, $i,j = 1\ldots p$ and a vectorial relative degree $r = [r_1,\ldots,r_p]$. Note that the output zeroing feedback $u = k_z(\xi,\eta)$ is unique [15] and is given by

$$u = k_z(\xi,\eta) = -A(\xi,\eta)^{-1}b(\xi,\eta) \tag{3}$$

with $b(\xi,\eta) = [b_1(\xi,\eta)\ldots b_p(\xi,\eta)]^T$.

For example, if the control system (1) is a single-input-single-output system with $f, G, h$ sufficiently smooth and if the relative degree is well-defined, then a local change of coordinates exists that transforms the control system into the given form. The multi-input-multi-output case is more involved [15]. However, control systems that are minimum phase in the sense of Byrnes-Isidori exhibit stable behavior under the constraint that the output is identically zero. More precisely:

*Definition 1:* The control system (1) under the Assumption 1 is said to possess the minimum phase property with respect to the equilibrium point $x = 0$, if $x = 0$ is asymptotically stable under the constraint $y(t) = 0$, $t \geq 0$. In other words, the zero dynamics

$$\dot{\eta} = q(0,\eta) - P(0,\eta)A(0,\eta)^{-1}b(0,\eta) \tag{4}$$

of the control system (1), respectively (2), are asymptotically stable at $\eta = 0$.

Further definitions and notations. A function $V:\mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(0) = 0$, $V(x) > 0$ for all nonzero $x$. $V$ is called radially unbounded, if $V(x) \to \infty$ whenever $\|x\| \to \infty$. A continuously differentiable, positive definite, radially unbounded function $V$ is called a Lyapunov function candidate. For a function $V:\mathbb{R}^n \to \mathbb{R}$, the row vector $\frac{\partial V}{\partial x}(x) = \nabla V(x) = [V_{x_1}(x)\ldots V_{x_n}(x)]$ denotes the derivative of $V$ with respect to $x$. Furthermore, $P > 0$, ($P \geq 0$) indicates that a symmetric matrix $P = (p_{ij})$, is positive (semi)definite.

## III. A Dissipation Inequality for the Minimum Phase Property

In the following, the main results of the paper are derived. In particular, a characterization of the minimum phase property for the control system (1) under the Assumption 1 is given in terms of a dissipation inequality, Theorem 1, and an additional smoothness result for the dissipation inequality, Theorem 2, is derived. To define the dissipation inequality for the minimum phase property, the following so-called derivative array (cf. e.g. [13]) is used:

*Definition 2:* The derivative array $H_r : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^{r_1+\ldots+r_p+p}$ of the output function $y = h(x) = [h_1(x) \ldots h_p(x)]^T$ in (1) is defined by the first $r$ Lie-derivatives of the output, i.e.,

$$H_r(x,u) = \begin{bmatrix} h_1(x) \\ \dot{h}_1(x) \\ \vdots \\ h_1^{(r_1)}(x) \\ h_2(x) \\ \vdots \\ h_p^{(r_p)}(x) \end{bmatrix} \tag{5}$$

with $\dot{h}_i(x) = \frac{\partial h_i}{\partial x}(x)(f(x)+G(x)u) = L_f h_i(x) + L_G h_i(x) u$ etc., i.e., the Lie-derivatives of $h_i$ with respect to (1) up to degree $r_i$. Notice that $H_r$ is a function of $x$ and $u$, since $h_i^{(r_i)}$ depends on $u$.

Using the derivative array, the first result in this section is a characterization of the minimum phase property in terms a dissipation inequality.

*Theorem 1:* The control system (1) under the Assumption 1 has the minimum phase property according to Definition 1 if and only if there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $\rho : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{r_1+\cdots+r_p+p}$ such that the dissipation inequality

$$\nabla V(x)(f(x)+G(x)u) < H_r(x,u)^T \rho(x,u) \tag{6}$$

is satisfied for all $u$ and all nonzero $x$ in a neighborhood of $x = 0$.

*Proof:* The first part of the proof of Theorem 1 shows an explicit construction of the functions $V, \rho$, in case the control system (1) is minimum phase. The second part shows that if the dissipation inequality (6) is satisfied, then the minimum phase property follows.

Part 1 ((1) is minimum phase $\overset{(2)}{\Rightarrow}$ (6) is satisfied): In the following, it is assumed that the control system (1) is represented in the input-output normal form (2). Since (1) is minimum phase, the zero dynamics of (1) is asymptotically stable and is given by

$$\dot{\eta} = q(0,\eta) - P(0,\eta)A(0,\eta)^{-1}b(0,\eta) = z(0,\eta), \tag{7}$$

which follows by substituting the output zeroing feedback

$$k_z(\xi,\eta) = -A(\xi,\eta)^{-1}b(\xi,\eta) \tag{8}$$

into (2) and by setting $\xi = 0$. Let $W$ be a continuously differentiable Lyapunov function of (7). The existence of such an Lyapunov function is guaranteed due to Massera's converse Lyapunov theorem [20], [28], [16]. Massera's theorem assumes a locally Lipschitz right-hand side of the differential equation for the existence of a smooth differential Lyapunov function. Since this is assumed in Assumption 1, $W$ exists. Define now a Lyapunov function candidate

$$V(\xi,\eta) = U(\xi) + W(\eta), \tag{9}$$

where $U$ is an arbitrary Lyapunov function candidate, i.e., a positive definite, radially unbounded, continuously differentiable scalar-valued function. The derivative of $V$ along the trajectories of (2) is given by:

$$\dot{V}(\xi,\eta) = \nabla U(\xi)\dot{\xi} + \nabla W(\eta)\dot{\eta}. \tag{10}$$

Next, two cases are distinguished: *Case 1:* $H_r$ is zero in (6), i.e., $\xi_1^i = \ldots = \xi_{r_i}^i = \dot{\xi}_{r_i}^i = 0$, $(u = k_z(\xi,\eta))$, $i = 1 \ldots p$. In this case define the value of $\rho$ to be zero, i.e.,

$$\rho(\xi,\eta,u) = 0. \tag{11}$$

What remains to show that (6) is satisfied is that $\nabla W(\eta)\dot{\eta} < 0$ holds for some neighborhood around $\eta = 0$. But this is the case, since asymptotic stability of the zero dynamics is assumed. *Case 2:* $H_r$ is not zero in (6), i.e., there exists $\xi_j^i \neq 0$ or $\dot{\xi}_{r_i}^i \neq 0$ $(u \neq k_z(\xi,\eta))$. In this case define the value of $\rho$ such that

$$\begin{aligned} \rho(\xi,\eta,u) &= H_r(\xi,\eta,u) \cdot \tilde{\rho}(\xi,\eta,u), \\ \tilde{\rho}(\xi,\eta,u) &> \frac{\nabla U(\xi)\dot{\xi} + \nabla W(\eta)\dot{\eta}}{\|H_r(\xi,\eta,u)\|^2}. \end{aligned} \tag{12}$$

The value of $\rho$ is finite since $H_r(\xi,\eta,u) \neq 0$. With the definition of $\rho$ by (11), (12) and $V$ according to (9), the dissipation inequality (6) is satisfied (in $(\xi,\eta)$-coordinates). Note that the dissipation inequality in the original coordinates can be obtained by the inverse transformation $x = \Phi^{-1}(\xi,\eta)$.

Part 2 ((6) is satisfied $\overset{(2)}{\Rightarrow}$ (1) is minimum phase): To show this, consider the zero dynamics, i.e., consider the dynamics which is defined by initial conditions $(\xi_0,\eta_0)$ with $\xi_0 = 0$ and by the output zeroing feedback $u = k_z(\xi,\eta)$. Under these initial conditions and under the output zeroing feedback $u = k_z(\xi,\eta)$, $y(t) = 0$ for all $t \geq 0$. Hence $H_r(0,\eta(t),u(t)) = 0$, $t \geq 0$ because of $\xi(t) = 0$, $t \geq 0$. Thus the dissipation inequality (6) turns into $\dot{V}(0,\eta(t)) < 0$ and therefore $V$ is a Lyapunov function and the equilibrium point $\eta = 0$ of the zero dynamics is asymptotically stable. ■

Theorem 1 establishes a symmetric statement between the minimum phase property and a dissipation inequality (6). To understand the dissipation inequality (6) is not difficult. However, a few points have to be explained. Firstly, the role of the derivative array $H_r$: The zero dynamics is the dynamics such that the output is identically zero. This dynamics evolves on the zero dynamics manifold, which is implicitly defined by $\|H_r(x,u)\| = 0$, since $H_r$ is identically zero, if $y(t) = 0$,

$t \geq 0$. Remember that $\rho$ in the proof of Theorem 1 has the form $\rho(\xi,\eta,u) = H_r(\xi,\eta,u)\tilde{\rho}(\xi,\eta,u)$, which turns the inequality (6) into

$$\nabla V(x)(f(x)+G(x)u) < \|H_r(x,u)\|^2\tilde{\rho}(x,u). \quad (13)$$

Therefore, stability on the manifold $\|H_r(x,u)\| = 0$ has to be studied, i.e., a Lyapunov function is needed subject to the constraint $\|H_r(x,u)\| = 0$. $\|H_r(x,u)\| > 0$ is not of interest. This situation is compactly expressed in inequality (6) ((13)), where $\rho$ plays the role of a penalization function. Geometrically speaking, inequality (6) guarantees negative definiteness of the derivative of $V$ only on a subset, namely on the set where $\|H_r(x,u)\| = 0$. For $\|H_r(x,u)\| > 0$, one can find a function $\rho$ such that the left side is dominated by the right side of the dissipation inequality (6). Algebraically speaking, the right side of the dissipation inequality (6) is the ideal generated by $\|H_r(x,u)\|$, i.e, the left side is negative definite modulo $\|H_r(x,u)\| > 0$.

Summarizing, the main ingredients to arrive at the dissipation inequality (6) are the so-called derivative array, which defines the hidden constraints and which finally defines the zero dynamics manifold, as well as a penalization argument, a well-known argument from optimization theory. Furthermore, in contrast to the ISS-like minimum phase characterization introduced in [19], Theorem 1 is necessary and sufficient to express the minimum phase property as defined by Byrnes and Isidori. Hence, Theorem 1 represents a complete characterization of the minimum phase property. It is also worthwile to remark that from the dissipation inequality (6) it can be very clearly seen that the notion of the minimum phase property is feedback invariant, since it must hold for all $u$.

In Theorem 1, no statement is made about the degree of smoothness of $\rho$. Even no statement on the existence of a continuous $\rho$ is made. However, for computational purposes for example, a guarantee of the existence of a smooth $\rho$ would be desirable. The next theorem shows that under Assumption 1, there exists indeed a smooth $\rho$. In particular the following proof of Theorem 2 is constructive and an explicit function $\rho$ is constructed. Due to simplicity of exposition, the construction is carried out for the case $p = 1$, i.e., Theorem 2 is stated for the single-input-single-output case. The construction for the multi-input-multi-output is more tedious but goes along the same lines.

*Theorem 2:* If the control system (1) under the Assumption 1 with $p = 1$ has the minimum phase property according to Definition 1, then there exists a smooth Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ and a smooth function $\rho : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{r+1}$ such that the dissipation inequality (6) is satisfied for all $u$ and all nonzero $x$ in a neighborhood of $x = 0$.

*Proof:* As in the proof of Theorem 1, it is assumed that the control system (1) is represented in the input-output normal form (2). Smoothness of the Lyapunov function $W$ for the zero dynamics (7) follows from Massera's converse Lyapunov theorem [20], [28]. In particular, it is assumed in Assumption 1 that the zero dynamics is sufficiently smooth, hence a sufficiently smooth $W$ exists. To show that also $\rho$ is sufficiently smooth, a smooth Lyapunov function candidate of the form

$$V(\xi,\eta) = \frac{1}{2}\xi^T\xi + W(\eta) \quad (14)$$

is chosen, i.e., $U(\xi) = \frac{1}{2}\xi^T\xi$ in (9). Hence, the dissipation inequality (6) for the control system (2), with $p = 1$, is given by:

$$\begin{aligned}
&\xi_1\xi_2 + \ldots + \xi_{r-1}\xi_r \\
&+\xi_r(b(\xi,\eta)+a(\xi,\eta)u) \\
&+\nabla W(\eta)(q(\xi,\eta)+p(\xi,\eta)u) \\
&< \xi_1\rho_1(\xi,\eta,u) + \ldots + \xi_r\rho_r(\xi,\eta,u) \\
&+(b(\xi,\eta)+a(\xi,\eta)u)\rho_{r+1}(\xi,\eta,u)
\end{aligned} \quad (15)$$

with $\xi = [\xi_1 \ldots \xi_r]^T$. In a first step, $\rho$ is chosen as

$$\begin{aligned}
\rho_i(\xi,\eta,u) &= \xi_{i+1} + \tilde{\rho}_i(\xi,\eta,u), \\
\rho_r(\xi,\eta,u) &= (b(\xi,\eta)+a(\xi,\eta)u) + \tilde{\rho}_r(\xi,\eta,u)
\end{aligned} \quad (16)$$

$i = 1 \ldots r-1$, with $\tilde{\rho}_i$ as new auxiliary functions. Hence (15) turns into

$$\begin{aligned}
&\nabla W(\eta)(q(\xi,\eta)+p(\xi,\eta)u) \\
&< \xi_1\tilde{\rho}_1(\xi,\eta,u) + \ldots + \xi_r\tilde{\rho}_r(\xi,\eta,u) \\
&+(b(\xi,\eta)+a(\xi,\eta)u)\rho_{r+1}(\xi,\eta,u).
\end{aligned} \quad (17)$$

In a second step, $u$ is replaced by

$$u = -\frac{1}{a(\xi,\eta)}(b(\xi,\eta)+v) \quad (18)$$

with $v$ as a new input. Therefore, one obtains from (17)

$$\begin{aligned}
&\nabla W(\eta)\left(q(\xi,\eta) - \frac{p(\xi,\eta)}{a(\xi,\eta)}(b(\xi,\eta)+v)\right) \\
&< \xi_1\tilde{\rho}_1(\xi,\eta,v) + \ldots + \xi_r\tilde{\rho}_r(\xi,\eta,v) + v\rho_{r+1}(\xi,\eta,v).
\end{aligned} \quad (19)$$

Due to the substitution (18), inequality (19) can be satisfied if and only if (17) can be satisfied. In a next step, $\rho$ is chosen as

$$\rho_{r+1}(\xi,\eta,v) = -\nabla W(\eta)\frac{p(\xi,\eta)}{a(\xi,\eta)} \quad (20)$$

and after rewriting (19), one arrives at

$$\begin{aligned}
&\nabla W(\eta)z(0,\eta) + \nabla W(\eta)\left(z(\xi,\eta) - z(0,\eta)\right) \\
&< \xi_1\tilde{\rho}_1(\xi,\eta,v) + \ldots + \xi_r\tilde{\rho}_r(\xi,\eta,v),
\end{aligned} \quad (21)$$

where the expression that corresponds to the zero dynamics is given by

$$z(\xi,\eta) = q(\xi,\eta) - p(\xi,\eta)\frac{b(\xi,\eta)}{a(\xi,\eta)}. \quad (22)$$

Since the control system is assumed to be minimum phase, the inequality

$$\nabla W(\eta)z(0,\eta) < 0 \quad (23)$$

holds locally. Thus, it is sufficient to show that

$$\begin{aligned}
&\nabla W(\eta)\left(z(\xi,\eta) - z(0,\eta)\right) + \xi^T\xi \\
&\leq \xi_1\tilde{\rho}_1(\xi,\eta,v) + \ldots + \xi_r\tilde{\rho}_r(\xi,\eta,v)
\end{aligned} \quad (24)$$

can be satisfied. Assumption 1 implies that the function $z$ is sufficiently smooth and therefore continuously differentiable. By applying a mean-value theorem for vector-valued functions, the so-called Hadamard lemma [25], [3], the difference $z(\xi,\eta)-z(0,\eta)$ can be written as

$$z(\xi,\eta)-z(0,\eta)=Z(\xi,\eta)\xi \tag{25}$$

with a continuous (smooth) matrix-valued function $Z$ defined by

$$Z(\xi,\eta)=\int_0^1 \frac{\partial z}{\partial x}((1-\theta)\xi,\eta)d\theta. \tag{26}$$

Hence, the inequality (24) can be written as

$$\begin{aligned}&\nabla W(\eta)Z(\xi,\eta)\xi+\xi^T\xi\\ \leq\ &\xi_1\tilde{\rho}_1(\xi,\eta,v)+\ldots+\xi_r\tilde{\rho}_r(\xi,\eta,v),\end{aligned} \tag{27}$$

from which the smooth functions $\tilde{\rho}_i$ easily follow such that (27) holds. For example, choose the $\tilde{\rho}_i$'s such that

$$\begin{aligned}&\nabla W(\eta)Z(\xi,\eta)+\xi^T\\ =&[\tilde{\rho}_1(\xi,\eta,v)\ldots\tilde{\rho}_r(\xi,\eta,v)].\end{aligned} \tag{28}$$

Therefore, inequality (24) is satisfied and thus also the desired dissipation inequality (15). Finally, note again that the dissipation inequality in the original coordinates can be obtained by the inverse transformation $x=\Phi^{-1}(\xi,\eta)$ of the input-output normal form transformation. ■

Notice that the converse statement of Theorem 2 follows immediately from the proof of Theorem 1. Furthermore, Definition 1 and all established results are of local nature with respect to the equilibrium point $x=0$. From the proofs of Theorem 1 and 2, however, global results can be easily established. Moreover, the results in this paper can be easily extended to the more general input-output normal form in [15] (p.310). In particular, this is one advantage of the established dissipation inequality since it is in principle also applicable to general control systems, as summarized next.

*Remark 1:* The affine structure of the control system (1) can be easily replaced in (6) by a general, nonaffine control system, i.e.,

$$\nabla V(x)f(x,u)<H_r(x,u)^T\rho(x,u), \tag{29}$$

which leads to a possible extension of the minimum phase property to nonaffine control systems. In this case, however, the zero dynamics might not be well-defined and the output zeroing feedback is not unique anymore. For generalized notions for the minimum phase property to control systems that are not affine in the control input, one may also consult [22], [19]. Since the minimum phase property is basically a matter of stability on manifolds, one may consult [9] which provides a general Lyapunov-based approach for such questions.

Summarizing, in this section a new characterization of the minimum phase property for control systems which posses a well-defined input-output normal form is derived. Moreover, the established characterization is suitable for computational purposes (cf. Example 2, Section V) and also applicable to control systems where an input-output normal form does not exist or where the relative degree is not well-defined (cf. Example 1, Section V).

## IV. A Connection to Performance Limitations

It is well-known that the minimum phase property is an important notion for describing fundamental performance limitations in feedback design. In particular, well-known is the Bode integral of the inverse sensitivity function (Bode T-integral), which relates the minimum phase property with limitations in tracking problems [23], [18], [11], [10], [21], [26], [27], [2], [29], [1]. One may ask the question, in how far the derived dissipation inequality in Section III reflects this fact. The discussion of this question is the subject of the current section.

Recently, in the interesting work by Seron, Braslavsky, Kokotović, and Mayne [27], Bode integrals were derived and interpreted using cheap control. In particular, the starting point in [27] was the following LQR problem

$$J_{\tilde{\varepsilon}}^{lq}=\inf_u \ \ \|y(t)\|_{L_2}^2+\tilde{\varepsilon}\,\|u(t)\|_{L_2}^2 \tag{30}$$

subject to the linear time-invariant control system

$$\dot{x}=Ax+Bu \tag{31}$$

$$y=Cx, \tag{32}$$

where $A,B,C$ are matrices of appropriate dimensions. By studying the solution when $\tilde{\varepsilon}\to 0$, i.e., the cheap control solution, a connection to the Bode T-integral was obtained under the assumption:

*Assumption 2:* The control system (31),(32) has relative degree one, all zeros are nonminimum phase, i.e., they lie in the open right half complex plane, and the control system is stabilizable and detectable.

Under these assumptions, the linear time-invariant control system (31),(32) can be always transformed into the input-output normal form with

$$A=\begin{bmatrix}A_1 & A_2\\ B_0 & A_0\end{bmatrix},\ B=\begin{bmatrix}B_1\\ 0\end{bmatrix}, \tag{33}$$

$A_0,A_1,A_2,B_1,B_0$ are matrices of appropriate dimension, and

$$y=Cx=\begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\xi\\ \eta\end{bmatrix}, \tag{34}$$

where $\xi\in\mathbb{R}^q$, $\eta\in\mathbb{R}^{n-q}$, $u\in\mathbb{R}^q$. Due to Assumption 2, $rank(B_1)=p$ and $-A_0$ is Hurwitz, i.e., all eigenvalues $\alpha_i$, $i=1\ldots n-p$ of $A_0$ have positive real part, since $A_0$ defines the zero-dynamics. Moreover, since the control system (31) is stabilizable, $(A_0,B_0)$ is controllable [27].

In the following a different viewpoint than the one in [27] is taken into account to relate the minimum phase property to the Bode T-integral by utilizing the dissipation inequality (6) as a minimum phase constraint in a certain optimization problem. Hence, the dissipation inequality (6) allows an interpretation in terms of fundamental performance limitations and Bode integrals. The proposed viewpoint, at the end, ties up to cheap control. However, the authors believe that the proposed viewpoint fits very natural when dealing with dissipation inequalities and performance limitations. In particular, the derivation of the Bode T-integral differs in two main points from the derivation presented in [27], as is explained in the following. The first difference is the problem formulation. Namely, the starting point is motivated by the following loosely formulated question: "How far is a nonminimum phase control system away from the minimum phase property?" More precisely, since the minimum phase property of a control system is governed by the placement of the sensors and actuators, the question to be asked is: Which one is the "closest" (fictitious) minimum phase output, let's say $y_{mp}$, with respect to an (actual) nonminimum phase output $y$ such that the control system (31) with the new output $y_{mp}$ defined by

$$y_{mp} = C_{mp}x = [I\ C_0] \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{35}$$

$C_0 \in \mathbb{R}^{q \times n-q}$, becomes minimum phase? Notice that (35) is not in the most general form, yet general enough for the purpose of establishing a connection to the Bode T-integral, as will be shown below. To quantify the notion of "closest" output, a metric must be specified, which will be the $L_2$-norm as in a LQR problem. Motivated by this, the starting point of the proposed approach is the following optimization problem:

$$J_\varepsilon^{mp} = \inf_{y_{mp},u} \ \|y(t) - y_{mp}(t)\|_{L_2}^2 + \frac{1}{\varepsilon}\|y_{mp}(t) - r(t)\|_{L_2}^2$$
$$\text{s.t. (6) w.r.t. (31), (33), (35),} \tag{36}$$

where $r$ is a given reference signal and $\varepsilon$ is a positive constant. That is, minimize the $L_2$-distance of the output trajectories relative to the objective to keep $y_{mp}$ close to a reference signal $r$ and subject to the constraint that the control system becomes minimum phase. In simple terms, (36) performs a comparison between $y$, $y_{mp}$ with respect to a particular reference signal $r$. This is a reasonable objective function. Consider for example the situation if one is applying an input-output linearization approach to a nonminimum phase system in order to do reference tracking. One way to solve this reference tracking problem approximately is to search for a (fictitious) minimum phase output to be able to carry out an input-output linearization with the additional demand that the minimum phase output behaves similar as the (actual) nonminimum phase output.

In order to simplify the derivation, the reference $r$ is in the following assumed to be zero. The goal of the remainder of this section will be to show that the identity

$$J_0^{lq} = J_0^{mp} \tag{37}$$

holds, which finally allows to relate the optimization problem (36) to Bode's T-integral, in the same way as this can be done for the optimization problem (30). However, it should be emphasised that the main motivation behind (36) is to obtain an optimal minimum phase output and not an optimal feedback as in (30).

In a first step, a quadratic-linear Ansatz for $V$, $\rho$ for the dissipation inequality (6) with respect to (31),(33),(35) is used, which leads to the following constraint in (36):

$$2x^T Q \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 2 \begin{bmatrix} x \\ u \end{bmatrix}^T H_1^T R \begin{bmatrix} x \\ u \end{bmatrix} \tag{38}$$

with

$$\begin{aligned} V(x) &= x^T Q x > 0 \\ \rho(x,u) &= 2R \begin{bmatrix} x \\ u \end{bmatrix} \\ H_r(x,u) &= H_1 \begin{bmatrix} x \\ u \end{bmatrix}, \end{aligned} \tag{39}$$

where

$$H_1 = \begin{bmatrix} I & C_0 & 0 \\ A_1 + C_0 B_0 & A_2 + C_0 A_0 & B_1 \end{bmatrix}. \tag{40}$$

To avoid the explicit calculation of the optimal feedback in (36), the idea is to use duality theory from optimization. This allows to obtain the optimal minimum phase output without an explicit knowledge of the optimal $u$ in (36). This duality-based approach is the second main difference with respect to the approach proposed in [27]. In the current context, roughly speaking, the dual viewpoint describes the situation as "for *all* feedbacks $u$, the $L_2$-distance is *larger than* (or equal to), let's say an optimal number $J_\varepsilon^{mp}$", whereas a primal viewpoint would describes the situation as "there exists *no* feedback $u$, such that the $L_2$-distance is *less than* $J_\varepsilon^{mp}(u^\star) = J_\varepsilon^{mp}$". Hence, the objective of a dual optimization problem is to find the best lower bound of (36). Assume for the moment that the matrix $C_0$ in (36) is constant. Then, the dual objective of (36) with $r = 0$ is to find the "largest" positive semidefinite matrix $P$ such that

$$x(0)^T P x(0) \leq \|C_0\eta(t)\|_{L_2}^2 + \frac{1}{\varepsilon}\|y_{mp}(t)\|_{L_2}^2 \tag{41}$$

holds for all $x(0)$ and for all $u$, with $\|y(t) - y_{mp}(t)\|_{L_2} = \|C_0\eta(t)\|_{L_2}$. A dual viewpoint seems to provide a very natural approach in characterizing and calculating fundamental limitations. The point is that (41) holds for *all* $u$. Hence, whatever control law one may apply, the $L_2$-distance is larger or equal to $x(0)^T P x(0)$. Therefore, no control law can break this limitation and thus it is a fundamental limitation. Furthermore, a characteristic property of solving dual optimization problems is that a dual solution provides in general a lower bound of the value function but no information about the optimizer (optimal feedback). Even though this is in general a disadvantage, for performance limitations it is not of interest which feedback achieves the best performance but it is of interest how large (restrictive) the limitation is. And this information is obtained by a dual solution. A dual solution approach is very common in optimization and in particular in convex optimization. Recently, results from convex duality theory have been applied to linear control systems in [5], [4], [12]. In particular, in [5], [4] linear quadratic regulator theory has been interpreted from a primal and from a dual point of view using semidefinite programming duality. Since the optimization problem in (36) is a singular problem, i.e., the objective function does not depend on $u$, the results from [5], [4] are not directly applicable, as it is also the case in the cheap control setup (30) with $\tilde{\varepsilon} = 0$. However, the necessary changes are minor with respect to [5], [4] in order to be able to exploit the dual viewpoint in the current setup. This is outlined in the following, going along the lines of [5], [4]. Consider the linear time-invariant control system (31) with an initial condition $x_0 = x(0)$, $x = [\xi\ \eta]^T$. Suppose that for a positive semidefinite matrix $P \geq 0$ it holds that:

$$\frac{d}{dt}x(t)^T Px(t) \geq -\left\|C_0\eta(t)\right\|^2 - \frac{1}{\varepsilon}\left\|y_{mp}(t)\right\|^2 \tag{42}$$

for all $t \geq 0$ and for all $x$ and $u$ satisfying (31) and it holds that $\lim_{t\to\infty} x(t) = 0$. Inequality (42) is equivalent to

$$2x^T P(Ax + Bu) \geq -\eta^T C_0^T C_0 \eta - \frac{1}{\varepsilon} x^T C_{mp}^T C_{mp} x. \tag{43}$$

(42) obviously implies (41), i.e.,

$$\begin{aligned} x(0)^T Px(0) \leq & \int_0^\infty \eta(t)^T C_0^T C_0 \eta(t) dt \\ & + \frac{1}{\varepsilon} \int_0^\infty x(t)^T C_{mp}^T C_{mp} x(t) dt. \end{aligned} \tag{44}$$

Assume now that the matrix $C_0$ in (36) is constant, then for the optimal value of (36) it holds that:

$$J_\varepsilon^{mp}(x(0)) \geq x(0)^T Px(0). \tag{45}$$

In order to use duality when $C_0$ is an optimization variable (not constant), one can proceed as follows. First, without loss of generality, rewrite (36) as

$$\begin{aligned} \inf_{Q>0,R,C_0} \inf_u \ & \left\|y(t) - y_{mp}(t)\right\|_{L_2}^2 + \frac{1}{\varepsilon}\left\|y_{mp}(t) - r(t)\right\|_{L_2}^2 \\ & \text{s.t. } (38). \end{aligned} \tag{46}$$

Next, apply the outlined duality arguments above to the inner optimization problem only, i.e., to

$$\inf_u \ \left\|y(t) - y_{mp}(t)\right\|_{L_2}^2 + \frac{1}{\varepsilon}\left\|y_{mp}(t) - r(t)\right\|_{L_2}^2, \tag{47}$$

which has the dual

$$\begin{aligned} \sup_{P\geq 0} \ & x(0)^T Px(0) \\ & \text{s.t. } (43). \end{aligned} \tag{48}$$

Therefore, a lower bound for the optimization problem (36) with $r = 0$ can be obtained by solving the following optimization problem:

$$\begin{aligned} \inf_{Q>0,R,C_0} \sup_{P\geq 0} \ & x(0)^T Px(0) \\ & \text{s.t. } (43), (38). \end{aligned} \tag{49}$$

The next result provides the solution of the optimization problem (49) for the case $\varepsilon \to 0$. This corresponds to the situation of minimizing the distance between the minimum phase output and the nonminimum phase output with the main priority that $y_{mp}$ is close to the reference $r = 0$. Such a situation is of interest, for example in exact input-output linearization with perfect tracking, i.e., when $y_{mp}$ is used to stabilize the control system by zeroing $y_{mp}$ and one is interested in the behavior of the actual output $y$.

*Proposition 1:* For any $x(0) \in \mathbb{R}^n$, the optimal solution

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_0 \end{bmatrix} \tag{50}$$

of the optimization problem (49) for $\varepsilon \to 0$ is given by

$$P_0 A_0 + A_0^T P_0 = P_0 B_0 B_0^T P_0. \tag{51}$$

$$B_0^T P_0 = C_0 \tag{52}$$

$$P_0 > 0, P_1 = P_2 = 0. \tag{53}$$

*Proof:* The proof is split into four parts.

*Part 1. (Dual inequality constraint* (43)*).* Since the inequality constraint (43) must hold for all $u$ and $x = [\xi\ \eta]^T$, it follows that

$$PB = \begin{bmatrix} P_1 B_1 \\ P_2 B_1 \end{bmatrix} = 0, \tag{54}$$

which can be easily verified by writing (43) as a matrix inequality. Moreover, since $rank(B_1) = q$, it follows that $P_1 = P_2 = 0$, i.e., that

$$P = \begin{bmatrix} 0 & 0 \\ 0 & P_0 \end{bmatrix}. \tag{55}$$

*Part 2 (Change of variables).* In the second part, a change of variables is carried out with

$$\bar{x} = \begin{bmatrix} \bar{\xi} \\ \bar{\eta} \end{bmatrix} = Tx = \begin{bmatrix} I & C_0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{56}$$

Thus, (33), (34), (35), (40) turn into

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ B_0 & A_0 - B_0 C_0 \end{bmatrix} \\ \bar{B} &= B \\ \bar{C} &= \begin{bmatrix} I & -C_0 \end{bmatrix} \\ \bar{C}_{mp} &= \begin{bmatrix} I & 0 \end{bmatrix} \\ \bar{H}_1 &= \begin{bmatrix} I & C_0 & 0 \\ \bar{A}_1 & \bar{A}_2 & B_1 \end{bmatrix} \end{aligned} \tag{57}$$

with

$$\begin{aligned} \bar{A}_1 &= A_1 + C_0 B_0 \\ \bar{A}_2 &= -A_1 C_0 + A_2 - C_0 B_0 C_0 + C_0 A_0. \end{aligned} \tag{58}$$

Notice that this change of variable does not affect the solution of the optimization problem, i.e.,

$$x(0)^T Px(0) = \bar{x}(0)^T (T^{-1})^T P T^{-1} \bar{x}(0) \tag{59}$$

due to (55).

*Part 3 (Minimum phase inequality constraint* (38)*.)* After this change of variables, it can be verified by following the steps in the proof of Theorem 2 (cf. (14), (16), (20), (28)), that (38) is satisfied if and only if $C_0$ satisfies

$$\bar{Q}_0(A_0 - B_0C_0) + (A_0 - B_0C_0)^T\bar{Q}_0 < 0, \tag{60}$$

for a $\bar{Q}_0 > 0$. This condition obviously ensures the minimum phase property by inspecting the matrix $\bar{A}$ in (57). Notice that the Ansatz of $\bar{Q}$ follows from (14), i.e.,

$$\bar{Q} = T^TQT = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & \bar{Q}_0 \end{bmatrix} > 0. \tag{61}$$

Moreover, notice that $\bar{Q}_0 > 0$ (as well as $R$, $\bar{R}$ respectively) is not unique, which, however, does not cause any problems. Alternatively, one could add additional constraints, e.g. $\bar{Q}_0 < cI$, $c \in \mathbb{R}$, to ensure boundedness of these variables.

*Part 4 (Derivation of the optimal $P_0$.)* Taking into account the change of variables (56) as well as (55), the inequality constraint (43) turns into the following matrix inequality

$$\begin{bmatrix} 0 & B_0^TP_0 \\ P_0B_0 & P_0(A_0 - B_0C_0) + (A_0 - B_0C_0)^TP_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{\varepsilon}I & 0 \\ 0 & C_0^TC_0 \end{bmatrix} \geq 0. \tag{62}$$

By applying the Schur complement [7], observe first that (62) for $\varepsilon \to \infty$ implies $P_0 \to 0$. Notice that this behavior is reasonable, because if no attention is paid to the control objective, then a comparison of outputs becomes meaningless. On the other hand, observe that for $\varepsilon \to 0$ (62) holds if and only if

$$(A_0 - B_0C_0)^TP_0 + P_0(A_0 - B_0C_0) + C_0^TC_0 \geq 0 \tag{63}$$

holds. Now, by considering the inner optimization in (49) ("sup"), it can be verified that the optimal solution $P_0$ must satisfy

$$(A_0 - B_0C_0)^TP_0 + P_0(A_0 - B_0C_0) + C_0^TC_0 = 0 \tag{64}$$

for any $C_0$ such that $A_0 - B_0C_0$ is Hurwitz. Otherwise, in case

$$(A_0 - B_0C_0)^T\tilde{P}_0 + \tilde{P}_0(A_0 - B_0C_0) + C_0^TC_0 = D \tag{65}$$

with $D \neq 0$, $D \geq 0$, and because $A_0 - B_0C_0$ is Hurwitz, $\tilde{P}_0$ would always satisfy $\tilde{P}_0 \leq P_0$, i.e., $\tilde{P}_0 = P_0 - \Delta_1$, $\Delta_1 \geq 0$, $\Delta_1 \neq 0$, and thus there exists an $\tilde{\eta}(0)$ such that

$$\tilde{\eta}(0)^TP_0\tilde{\eta}(0) - \tilde{\eta}(0)^T\tilde{P}_0\tilde{\eta}(0) > 0. \tag{66}$$

This can be observed by substituting $\tilde{P}_0 = P_0 - \Delta_1$ into (65) with $P_0$ satisfying (64). Thus, the supremum is achieved if (64) holds. Next, it is shown that $C_0$ is given by

$$C_0 = B_0^TP_0. \tag{67}$$

This can be verified by considering the outer optimization in (49) ("inf"). Otherwise, in case $\tilde{C}_0 = B_0^T\tilde{P}_0 + \Delta_2$ with $\Delta_2 \neq 0$, one would get instead of (64)

$$\begin{gathered}(A_0 - B_0(B_0^T\tilde{P}_0 + \Delta_2))^T\tilde{P}_0 + \\ \tilde{P}_0(A_0 - B_0(B_0^T\tilde{P}_0 + \Delta_2)) + \\ (B_0^T\tilde{P}_0 + \Delta_2)^T(B_0^T\tilde{P}_0 + \Delta_2) = 0,\end{gathered} \tag{68}$$

which simplifies to

$$A_0^T\tilde{P}_0 + \tilde{P}_0A_0 - \tilde{P}_0B_0B_0^T\tilde{P}_0 + \Delta_2^T\Delta_2 = 0. \tag{69}$$

Let $\tilde{P}_0 = P_0 + \Delta_3$, where $P_0$ satisfies

$$A_0^TP_0 + P_0A_0 - P_0B_0B_0^TP_0 = 0. \tag{70}$$

Then, (69) turns into

$$(A_0 - P_0B_0B_0^T)^T\Delta_3 + \Delta_3(A_0 - P_0B_0B_0^T) + \Delta_2^T\Delta_2 = 0. \tag{71}$$

Because of (70), $A_0 - P_0B_0B_0^T$ is Hurwitz and (71) has a positive semidefinite solution $\Delta_3 \neq 0$. Therefore $\tilde{P}_0 \geq P_0$ and thus there exists an $\tilde{\eta}(0)$ such that

$$\tilde{\eta}(0)^T\tilde{P}_0\tilde{\eta}(0) - \tilde{\eta}(0)^TP_0\tilde{\eta}(0) > 0. \tag{72}$$

In other words, the infimum is achieved with $C_0 = B_0^TP_0$, where $P_0$ is the solution of (70). Notice that a unique $P_0 > 0$ exists because $(A_0, B_0)$ is controllable and $-A_0$ is Hurwitz. ■

In the following, parts of the results presented in [27] are shortly summarized in order to establish a connection to the Bode T-integral. In [27], cheap control was used to interpret performance limitations and Bode integrals from a new angle. In particular, it has been shown that the value of the Bode T-integral, i.e.,

$$\frac{1}{\pi}\int_0^\infty log|T(j\omega)|\frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \sum_i \frac{1}{\alpha_i} \tag{73}$$

with $\alpha_i$ being the right half plane zeros of $T$ and $K_v = \frac{1}{T(0)}\lim_{s\to 0}\frac{dT(s)}{ds}$ is the velocity constant, is equivalent to the cost of the cheap control problem (30), where the penalization $\tilde{\varepsilon}$ of the control effort tends to zero. In particular, for a stable inverse sensitivity transfer function $T$, $T(0) = 1$, it was shown that

$$\frac{1}{2}\int_0^\infty e(t)^2dt = \sum_i \frac{1}{\alpha_i} \tag{74}$$

with $e(t) = y(t) - r$ is the error in transferring a single-input-single-output linear control system from rest to the setpoint $r = 1$ using the cheap optimal control (consult [27] for more details). The important point is now that the Riccati equation (51) also appears in exactly the same form in cheap control. Even more, the Riccati equation (51) is a main building block in [27] (cf. eq. (9) and (15)) which allows to establish the connections to the Bode integrals. In particular:

*Proposition 2:* The solution of the optimization problem (49) respectively (36) allows to establish a link to Bode's T-integral for $\varepsilon \to 0$, since the Riccati equation (51) also appears in cheap control.

*Proof:* See [27]. Note that from [27] also follows that the lower bound obtained in (49) is indeed the optimal solution of (36) (follow the arguments below, i.e., eq. (76)). Alternatively, this can be seen by the fact that the optimization problem (47) is convex. Hence, the dual solution is tight (cf. [4], [5]). ■

To demonstrate Proposition 2, the following control problem is considered. The objective is to design a high-gain feedback for the linear time-invariant system (31) to achieve fast reference tracking for step inputs. Since the control system (31) with the output (34) is nonminimum phase, no high-gain feedback design can be carried out. However, by solving the optimization problem (36) for $\varepsilon \to 0$, a new output (35) is obtained such that the control system (31), together with the new output (35), $C_0$ defined by (52), has relative degree one and is minimum phase, and thus a high-gain feedback design can be carried out. In particular, for a single-input-single-output control system, with $B_1 > 0$, one could apply a high-gain output feedback of the form

$$u = r - \frac{1}{\epsilon} y_{mp} \tag{75}$$

with $\epsilon$ as (positive) gain and $r$ as a reference signal. The high-gain feedback loop is depicted in Figure 1. Then the question which arises is: Is the new output $y_{mp}$ the best choice that can be used instead of $y$ to achieve the best possible tracking error in the $L_2$-sense, for a step input when one is applying the high-gain output feedback (75)? The answer is yes, since for any other minimum phase output $\tilde{y}_{mp}$ the following relation holds:

$$\begin{aligned} \frac{1}{2}\int_0^\infty (\tilde{y}_{mp}(t) - y(t))^2 dt \geq \\ \frac{1}{2}\int_0^\infty (y_{mp}(t) - y(t))^2 dt &= \sum_i \frac{1}{\alpha_i}, \end{aligned} \tag{76}$$

for $\epsilon \to 0$. This becomes clear, if one observes that the applied high-gain feedback (75) is the feedback obtained by cheap control (compare [27], Eq. (13)). In particular, the high-gain feedback ensures essentially the identity $y_{mp}(t) \equiv r(t)$. Thus, in the current scenario, the optimal value of (36) is

$$\lim_{\varepsilon \to 0} \frac{1}{2} J_\varepsilon^{mp} = \frac{1}{2} \inf_{y_{mp}, u} \|y(t) - y_{mp}(t)\|_{L_2} = \sum_i \frac{1}{\alpha_i}, \tag{77}$$

which establishes a connection to the Bode T-integral. A numerical example is discussed in Section V, Example 3.

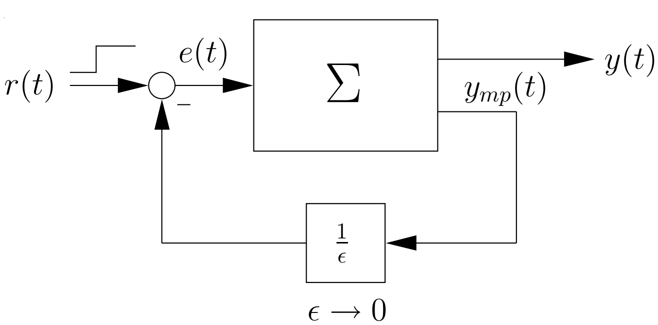


Fig. 1. High-gain output feedback with the optimal minimum phase output for following step inputs.

Summarizing, a different viewpoint with respect to [27] was used to connect performance limitations and the minimum phase property based on the characterization introduced in Section III. Even though most of the arguments in the last part of this section are based on [27], the first part gives new insight and is based on the idea to search for a new minimum phase output which is closest to the nonminimum phase output in the $L_2$-sense. Moreover, instead of using classical linear quadratic regulator theory, semidefinite programming duality has been exploited to establish a lower bound for fundamental performance limitations. The derivation in this section might be considered as too cumbersome, however, there are three interesting properties that make it worthwile do to it in this way. First, as already pointed out in this section, the dual point of view is a very natural viewpoint with respect to fundamental performance limitations. This is compactly expressed in the fact that the inequality constraints in the optimization problem (49) depends on $u$ (cf. (43), (38)) and that it must hold for all $u$. Hence, clearly no particular control law can do better and no optimal control law is needed for the analysis, i.e., the optimal feedback does not appear explicitly in the dual approach. Second, the optimization problem (49) is set up as an open-loop problem (searching for optimal outputs) whereas cheap control is set up as a closed-loop problem (searching for optimal feedback). That an open-loop setting may give additional inside is demonstrated in Example 3, Section V. Third, the constraints in the optimization problem (49) are nothing else than dissipation inequalities. The first dissipation inequality constraint in (49) ensures that one obtains a lower bound of the fundamental performance limitation, which results from (42). The second dissipation inequality constraint in (49) ensures the minimum phase property which results from (6). A formulation in terms of dissipation inequalities has an important advantage because it allows, at least formally, to formulate fundamental performance limitations for general nonlinear control systems where a normal-form must not exists or the relative degree need not be well-defined.

## V. Examples

In the following, three examples are given which illustrate the results in the previous sections. In particular, attention is paid to the following three aspects: computability, generalizability, and performance limitations.

### *Example 1*

This example illustrates that the dissipation inequality (6) can also be applied in case the control system does not have a well-defined relative degree and is not affine in the input. Consider the nonaffine control system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 e^u \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_2 u \\ y &= x_2, \end{aligned} \tag{78}$$

which has relative degree two except for $x_2 = 0$. Applying (6) with $V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$ yields

$$\begin{aligned} &-x_1^2 + x_1 x_3 e^u + x_2 x_3 + x_2 x_3 u \\ &< x_2 \rho_1(x,u) + x_3 \rho_2(x,u) + x_2 u \rho_3(x,u). \end{aligned} \tag{79}$$

For example, by choosing $\rho_1(x,u) = x_3 + x_2, \rho_2(x,u) = x_1 e^u + x_2 u + x_3, \rho_3(x,u) = 0$, one obtains $-x_1^2 - x_2^2 - x_3^2 < 0$. Thus global asymptotic stability of the zero dynamics is established, i.e., the control system (78) is (globally) minimum phase.

*Example 2*

This example illustrates that the dissipation inequality (6) for the minimum phase property is particularly suited for a minimum phase test for control systems with polynomial nonlinearities. In general, it is very difficult to search for a Lyapunov function $V$ and a function $\rho$ such that (6) holds However, recently established methods from computational real algebraic geometry based on semidefinite programming and the sum of squares decomposition allow to verify the dissipation inequality (6) very efficiently in case all the functions involved are of polynomial type (consult for example [24], [14] and references therein). The following example demonstrates this fact, without going into the computational details. Consider the control system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1 x_2 + x_1 x_2^2 u \\ \dot{x}_2 &= -x_2 + x_4 - x_1 x_4 \\ \dot{x}_3 &= x_1^2 + x_2 + x_3 + u \\ \dot{x}_4 &= -x_4 + x_1 x_2 + x_1 x_2 x_3 \\ y &= x_3, \end{aligned} \tag{80}$$

which has relative degree one. By using semidefinite programming and sum of squares techniques, the following quadratic Lyapunov function

$$\begin{aligned} V = & \, 5.11 x_1^2 + 3.82 x_2^2 - 0.31 x_2 x_3 \\ & + 2.35 x_3^2 - 0.07 x_1 x_3 + 0.07 x_2 x_4 \\ & - 1.26 x_3 x_4 + 4.94 x_4^2 \end{aligned} \tag{81}$$

was found. Furthermore, a function $\rho$ was found with monomials of degree one to four. Therefore, it was possible to prove in a computationally efficient way that the control system (80) is globally minimum phase.

*Example 3*

This example illustrates the performance limitation caused by unstable zeros, as discussed in Section IV. Consider the linear control system

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta_1 \\ \eta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{82}$$

with

$$y = [1 \ 0 \ 0] \begin{bmatrix} \xi \\ \eta_1 \\ \eta_2 \end{bmatrix}. \tag{83}$$

The transfer function corresponding to (82), (83) is given by

$$G(s) = \frac{(s-1)^2}{s^3 - s^2 + 1}. \tag{84}$$

The given control system is unstable, controllable, observable, has relative degree one, and two unstable zeros at $\alpha_1 = \alpha_2 = 1$, i.e., the Bode T-integral has the value

$$\sum_i \frac{1}{\alpha_i} = 1 + 1. \tag{85}$$

Next, the optimal minimum phase output

$$y_{mp} = [1 \ c_1 \ c_2] \begin{bmatrix} \xi \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{86}$$

is computed using (52) and (51) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \; B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{87}$$

As a result, one obtains

$$c_1 = 0, \; c_2 = 4. \tag{88}$$

Therefore, the minimum phase transfer function of (82), (86), (88) is given by

$$G_{mp}(s) = \frac{(s+1)^2}{s^3 - s^2 + 1}. \tag{89}$$

The control system with the new output $y_{mp}$ is minimum phase and the zeros are mirrored from $+1$ to $-1$. Finally to illustrate the performance limitations, simulations are carried out in a high-gain feedback configuration as shown in Figure 1. Two simulation results for $\epsilon = 0.1$ and for $\epsilon = 0.001$ are shown in Figure 2 and in Figure 3. All states are initialized with zero and the reference input is given by a step of magnitude $r = 1 + \epsilon^{-1}$ in order to get $T(0) = G_{mp}(0)(1 + \epsilon^{-1} G_{mp}(0))^{-1} = 1$.

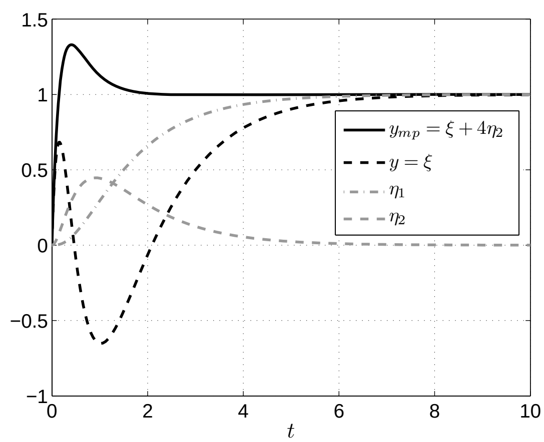


Fig. 2. Step response with a feedback gain of $\epsilon^{-1} = 10$.

One can see that for $\epsilon = 0.001$, the minimum phase output $y_{mp} = \xi + 4\eta_2$ follows almost immediately the step whereas the nonminimum phase output $y = \xi$ does not. Numerical integration of this discrepancy between $y$ and $y_{mp}$ yields the expected performance limitation caused by the unstable zeros

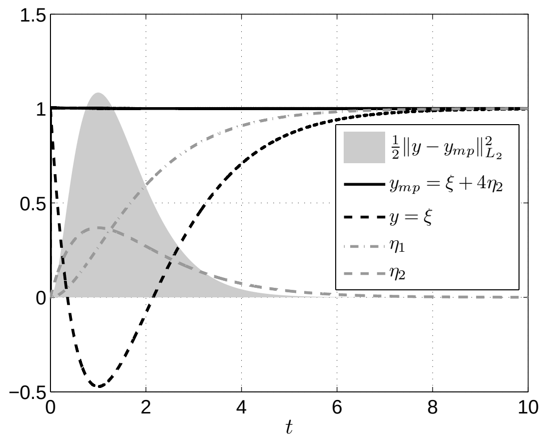


Fig. 3. Step response with a feedback gain of $\epsilon^{-1} = 1000$.

(see Figure 3, shaded area):

$$\begin{aligned} 2 =& \frac{1}{2}\int_0^\infty (y_{mp}(t) - y(t))^2 dt \\ =& 8\int_0^\infty \eta_2(t)^2 dt = \sum_i \frac{1}{\alpha_i} = 1 + 1. \end{aligned} \tag{90}$$

The starting point of the performance limitation consideration in Section IV was the question of asking for the minimal effort to reconfigurate the output measurements such that a nonminimum phase control system becomes minimum phase. This has been demonstrated in the current example.

*Remark 2:* Since the minimum phase property is a phase concept, a next natural step is to compare the Bode plots between $G$ and $G_{mp}$. And indeed, what can be immediately seen there is that

$$|G(i\omega)| = |G_{mp}(i\omega)|, \tag{91}$$

i.e., the gain is unchanged and a difference can be only observed in the phase. This observeration holds in general for single-input-single-output linear control systems, which follows from the so-called mirroring property [27], [18]. The mirroring property is the fact that the unstable zeros of $G$ are the mirror image of the stable zeros of $G_{mp}$. Thus, obviously

$$|G(s)| = \left|\frac{N(s)}{D(s)}\right| = \left|\frac{N_{mp}(s)}{D(s)}\right| \underbrace{\left|\frac{N(s)}{N_{mp}(s)}\right|}_{=1} = |G_{mp}(s)| \tag{92}$$

holds. This observation demonstrates nicely the fact that studying performance limitations in an open-loop setting (searching for minimum phase outputs) is convenient. In a closed-loop setting, such observation would not appear so clearly, except, of course, one would cut the loop open. As a final remark, notice that the difference between the phase of $G$, $G_{mp}$ is completely defined by the zeros (cf. e.g. [6], [31]).

## VI. Conclusions

The paper has three contributions. The first contribution is a characterization of the minimum phase property of nonlinear control systems in terms of a dissipation inequality. This allows to describe the notion of the minimum phase property, which was originally developed in nonlinear geometric control [15], with the help of a Lyapunov-based argument, in case the control systems possesses an input-output normal form. The main idea is the use of a so-called derivative array and a penalizing function, which allows to characterize the stability of the zero dynamics in terms of a dissipation inequality without an explicit knowledge of the equations that define the zero dynamics.

The second contribution of this paper shows that if the control system is sufficiently smooth, then the functions that appear in the derived dissipation inequality (6) can also be chosen smooth. Moreover, demonstrated on an example, it has been shown that the derived dissipation inequality that characterizes the minimum phase property is in particular suitable for a minimum phase analysis using efficient numerical algorithms. It has also been shown by an example that the dissipation inequality can be very easily applied to control systems that are not affine in the input and thus allow a way to generalize the notion of the minimum phase property very easily. Another advantage of the proposed dissipation inequality is the conceptual simplicity.

The third contribution is a different viewpoint on fundamental performance limitations caused by a nonminimum phase behavior, which allows to reveal connections to the Bode T-integral utilizing the derived minimum phase characterization. In particular, a dual viewpoint has been discussed, which asks, roughly speaking, for the closest output such that a nonminimum phase control system becomes minimum phase. Even though one may argue that this modified viewpoint is just a reinterpretation of well-known cheap control results, it seems that this open-loop viewpoint is interesting on its own and furthermore, the proposed dual, open-loop optimization problem is a natural alternative approach for the analysis of fundamental performance limitations in combination with the derived dissipation inequality (6). Furthermore, the presented derivation can be helpful in characterizing fundamental performance limitations for general nonlinear control systems, since the proposed optimization problem is based on dissipativity constraints, which can be formulated for general nonlinear control systems.

There are several interesting points for future reseach. Since the penalizing function $\rho$ is motivated from optimization theory, one can also consider $\rho$ as a Lagrange multiplier or as a dual variable. This may be of particular interest in connection with performance limitations and further investigation in this direction is needed. Second, instead of studying the minimal effort to reconfigurate the output sensor in order to turn a nonminimum phase control system into a minimum phase control system, one could also study the minimal effort to reconfigurate the input actuators or both sensors and actuators simultaneously. Finally, the dissipation inequality (6) is similar to a generalized phase or passivity condition [15], [17], [16], due to the appearence of the inner product in the dissipation inequality. This similarity may be useful to extend passivity-based results to minimum phase control systems with a higher relative degree.

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