Stability Analysis for Time-Delay Systems using Rekasius's Substitution and Sum of Squares

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Abstract— **In this paper, a new delay-dependent stability analysis for time-delay linear time-invariant (TDLTI) systems is derived. In contrast to many recent approaches, which often utilize Lyapunov-Krasovskii functionals and linear matrix inequalities, an alternative approach is proposed in this paper. The proposed stability analysis is formulated in the frequency domain and investigates the characteristic equation by using the socalled Rekasius substitution and recently established sum of squares techniques from computational semialgebraic geometry. The advantages of the proposed approach are that the stability analysis is often less conservative than many approaches based on Lyapunov-Krasovskii functionals, as demonstrated on a well-known benchmark example, and that the stability analysis is very flexible with respect to additional analysis objectives.**

I. INTRODUCTION

Stability analysis of time-delays systems has been systematically studied in systems and control theory since many decades [13], [21], [6], [5]. There are at least two main reasons for this continuously strong interest. The first reason is simply due to the fact that time-delay systems appear in a wide range of applications. For example, the range of applications spreads from traditional engineering applications, like conveyor belts, to modern engineering applications, like computational delays in real-time computation. Especially interesting applications are currently challenging areas like communication and information theory, where time-delays play an important role, for example when studying stability of controlled networks, internet, or guaranteed high-speed transmission protocols. The second reason for this continuously strong interest stems from the fact that there are still many unsolved problems in the stability analysis of time-delay systems. Even for time-delay linear time-invariant (TDLTI) systems, a complete stability analysis is still a difficult problem. One of the main reasons is that time-delay systems are infinite dimensional systems. This is especially difficult, if the stability analysis should be performed in a computationally efficient way. The objective of this paper is to establish such a computationally efficient delay-dependent stability analysis for TDLTI systems. In general, there are two basic methodologies to perform a stability analysis for TDLTI systems, namely frequency domain methods and time domain methods. Time domain methods are often used in the literature to perform a delay-dependent stability analysis. In particular Lyapunov-Krasovskii functionals and linear matrix inequalities are very popular and successful. This approach allows to circumvent the problems induced by the infinite dimensional character of time-delay systems and allows to use semidefinite programming algorithms. In particular, due to semidefinite programming, there has been a substantial increase of work in this area of research, e.g., [8], [12], [11], [3], [7], [25], [23], [4], [1].

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In this paper, a new stability analysis for TDLTI systems based on a frequency domain formulation is proposed. The objective is to obtain a lower bound for the maximal tolerated time-delay of a TDLTI system without jeopardizing stability. The proposed stability analysis exploits two concepts, namely the socalled Rekasius substitution, i.e., a bilinear transformation, and recently established sum of squares techniques from computational semialgebraic geometry. By using the Rekasius substitution, the transcendental characteristic equation can be transformed into an algebraic equation, i.e., into a polynomial equation. Based on this polynomial equation, a new lower bound for the maximal tolerated time-delay is derived in this paper. Furthermore, it is shown that sum of squares techniques can be used to compute this lower bound efficiently. In simple terms, sum of squares techniques allow to check if a polynomial in several variables is positive semidefinite. Analogous to linear matrix inequalities, sum of squares techniques are based on efficient and reliable semidefinite programming algorithms. Hence, the proposed stability analysis approach is based on the same efficient numerical tools as Lyapunov-Krasovskii functional approaches. One advantage of the new proposed stability analysis is that one obtains often better lower bounds than for example by using Lyapunov-Krasovskii functional approaches, which is demonstrated on a benchmark example in this paper. Another advantage is the gained flexibility due sum of squares techniques in order to include additional analysis objectives.

The remainder of the paper is structured as follows: In Section II, some preliminary facts on TDLTI systems and on the socalled Rekasius substitution are summarized. The basic idea of the new proposed stability analysis is motivated and explained in Section III. Section IV provides the mathematical justification of the proposed stability analysis and an efficient algorithm based on sum of squares techniques. Numerical examples are given in Section V and the results are compared with methods based on Lyapunov-Krasovskii functionals. Finally, a summary and an outlook is given in Section VI.

II. PRELIMINARIES

In this paper, the following problem is studied. Given the class of time-delay linear time-invariant (TDLTI) systems, i.e.,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $\tau \geq 0$ is the time delay, and $A_0, A_1 \in \mathbb{R}^{n\times n}$ are given constant matrices. Further, it is assumed that

$$A_0 + A_1 \text{ is Hurwitz}, \tag{2}$$

i.e., system (1) is asymptotically stable for $\tau = 0$. The objective is to find the maximal time delay τ_{max} such that the system (1) is asymptotically stable for all $\tau \in [0, \tau_{max}]$.

The basic idea of the proposed approach is to study the location of the roots of the characteristic equation using the socalled Rekasius substitution [19] and recently established sum of squares techniques [2], [17], [15] from computational semialgebraic geometry. The characteristic equation of the system (1) is given by

$$\Delta(s,\tau) = det\left(sI - A_0 - A_1 e^{-\tau s}\right), \tag{3}$$

or equivalently by

$$\Delta(s,\tau) = \sum_{k=0}^{n} a_k(s) e^{-k\tau s}, \tag{4}$$

where a_k are polynomials in s of degree $n-k$ with real coefficients. System (1) is asymptotically stable if and only if all roots of (3) respectively (4) lie in the open left half complex plane [5]. Unfortunately the characteristic equation is transcendental due to $e^{-k\tau s}$ and has in general infinitely many roots. This is the main reason why a stability analysis of TDLTI system is a nontrivial task. The first central building block in the proposed approach is the socalled Rekasius substitution [19], [6]. Rekasius introduced the following substitution in (3) respectively (4):

$$e^{-\tau s} = \frac{1-Ts}{1+Ts}, \tag{5}$$

$T \geq 0$, which is defined only on the imaginary axis in the complex plane, i.e., $s = i\omega$. However, equation (5) holds for $s = i\omega$ if and only if

$$\tau = \frac{2}{\omega}\left(tan^{-1}(\omega T) + k\frac{\pi}{2}\right), \tag{6}$$

$k = 0, 1, 2, \ldots$ Note that this substitution is a transformation (one-to-one) for $s = i\omega$, $\tau \geq 0$, $T \geq 0$. This fact is very well known and is studied in the theory of complex functions under the name bilinear or linear fractional or Möbius transformation. However, the crucial property of this substitution is that it turns the transcendental characteristic equation (3) respectively (4) into an algebraic (rational) equation. Interesting results based on the Rekasius substitution (also known as the socalled pseudodelay technique [13]) has been established for example in [14]. There, results have been derived which give answers to questions concerning the crossing properties of the roots of the characteristic equation with respect to the imaginary axis. However, in comparison to Lyapunov-Krasovskii functionals, rather less attention has been paid to this important transformation. Summarizing, the bottom line of Rekasius's substitution [19] is the fact that

Lemma 1: $s = i\omega$ ($\omega \geq 0$) is a root of the characteristic equation

$$det\left(i\omega I - A_0 - A_1 e^{-\tau i\omega}\right) = 0 \tag{7}$$

for a $\tau \geq 0$ if and only if

$$det\left(i\omega I - A_0 - A_1 \frac{1 - Ti\omega}{1 + Ti\omega}\right) = 0 \tag{8}$$

for a $T \in [0, \infty]$. The relation between τ and T is given by (6).

In other words, on the imaginary axis (and only on the imaginary axis), (7) has a root on it if and only if (8) has a root on it. Hence, (8) instead of (7) can be used to obtain the stability bound τ_{max} for the TDLTI system (1). Therefore, by using for example the Routh-Hurwitz criterion, the roots of the polynomial

$$D(s,T) = det\left((sI - A_0)(1+Ts) - A_1(1-Ts)\right) \tag{9}$$

on the imaginary axis can be located. Note that the polynomial equation (9) is obtained by applying Rekasius's substitution to (4) and by multiplying the resulting equation with $(1+Ts)^n$. The Routh-Hurwitz type approach was for example exploited in [24], [14]. Furthermore, note that the Routh-Hurwitz approach is exact, however, a disadvantage of the Routh-Hurwitz approach is that conditions (e.g., socalled Routh's array) depend now on T and hence one has essentially to compute all roots of (9), i.e., a polynomial in two variables, to finally obtain τ_{max}, which is computationally hard.

III. BASIC IDEA

The idea of this paper is to circumvent the computation of all roots of (9), which is shown in the following. Remember, it is assumed that for $\tau = 0$, system (1) is asymptotic stable. Hence for $\tau = 0$, the characteristic equation (3) has no roots on the imaginary axis. Therefore, by Lemma 1, (8) has no real positive solutions ω, T. In other words,

$$|D(i\omega, T = 0)| > 0 \tag{10}$$

for all positive ω. Now by increasing T, which corresponds to increasing τ due to (6), one expects that there exists a maximal T, i.e., T_{max} such that the roots of (3) cross the imaginary axis, and hence there exists a maximal T_{max} such that

$$|D(i\omega = i\omega_0, T = T_{max})| = 0 \tag{11}$$

for a certain ω_0, where $|.|$ denotes the absolute value. From this, one could suggest the following approach. In a first step, find the maximal T, i.e., T_{max} such that

$$|D(i\omega, T)| > 0 \tag{12}$$

holds for

$$\forall T \in [0, T_{max}], \forall \omega \in [0, \infty), \tag{13}$$

where T_{max} is found for example by a line search. In a second step, compute ω_0 in (11) and by using the function (6), compute a lower bound of τ_{max}, i.e., τ_0 given by

$$\frac{2}{\omega_0} tan^{-1}(\omega_0 T_{max}). \tag{14}$$

Let's discuss the first step in the above approach. The main idea is to replace the computation of the roots by a positivity condition. This is an advantage because condition (12) can be checked via efficient numerical methods, in particular by using sum of squares techniques. Hence, one can avoid to compute roots of polynomials. Now, let's discuss the second step in the above approach. The idea is rather obvious but unfortunately not correct. The fault in the arguments above is that there may exists another pair $(\tilde{\omega}_0, \tilde{T}_{max})$ with $\tilde{T}_{max} > T_{max}$ but with $\tilde{\tau}_0 < \tau_0$, since the relation between T and τ in function (6) depends also on ω and is therefore not monotonic. Summarizing, the first step bridges the connection to sum of squares techniques. The second step, however, does not lead to a lower bound for τ_{max}. Nevertheless, it is possible to modify the above approach, by keeping the positivity conditions, which finally leads to the main results of this paper in Section IV. The modified approach is as follows:

Stability Analysis Test: *Step 1: Compute the maximum (supremum) T_{max} and the minimal (infimum) ω_{min} such that*

$$|D(i\omega, T)| > 0 \tag{15}$$

holds for

$$\begin{aligned} &\forall T \in [0, T_{max}], \forall \omega \in [0, \infty), \\ &\forall T \in [0, \infty), \forall \omega \in [\omega_{min}, \infty). \end{aligned} \tag{16}$$

Step 2: Compute a lower bound of τ_{max} by

$$\underline{\tau_{max}} = \frac{2}{\omega_{min}} tan^{-1}(\omega_{min} T_{max}). \tag{17}$$

The only modification made is to replace (13) by a (16). Note that the strict inequality in (15) is crucial since $|D(i\omega, T)| \geq 0$. Furthermore, for the strict inequality $|D(i\omega, T)| > 0$, one will not achieve a maximum or minimum but a supremum or infimum. For computational purposes, however, $|D(i\omega, T)| \geq \varepsilon$, $\varepsilon > 0$ a small constant, will be used instead of (15). To see that this is indeed sufficient, i.e., to see that $\underline{\tau_{max}}$ is a lower bound of τ_{max}, one has to investigate the function (6). In the following, this is shown graphically and a proof is given in Section IV.

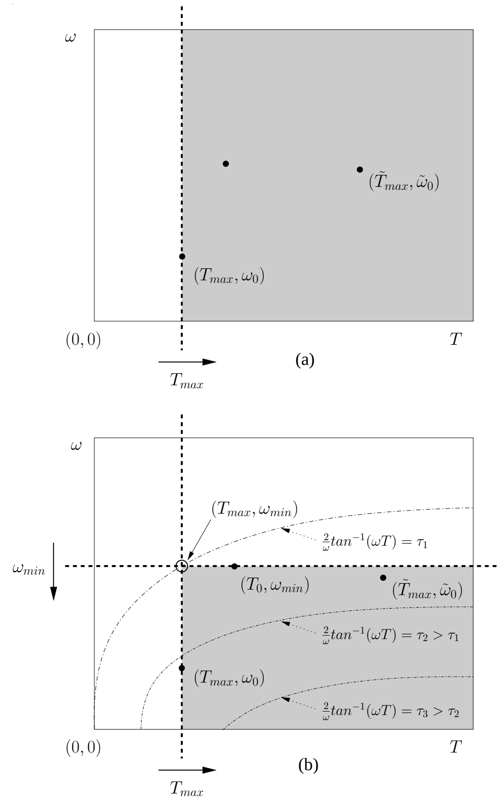

Fig. 1. Schematic representation of the $T - \omega$ plane.

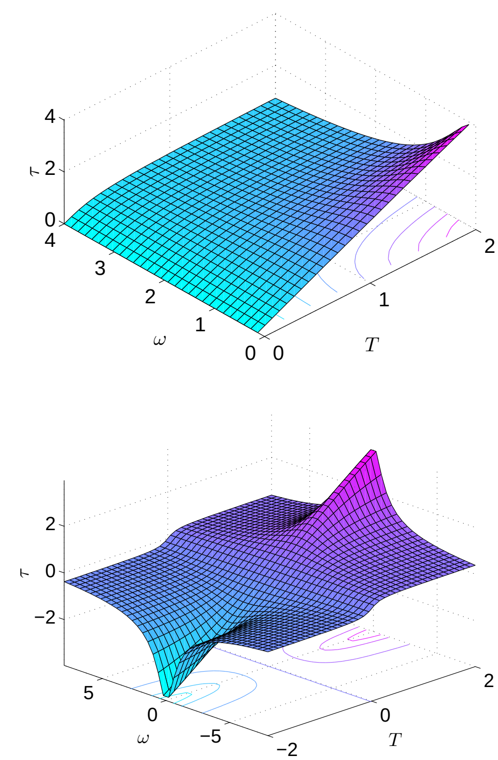

Fig. 2. Function (6) for $k = 0$.

Figure 1(a) shows basically the situation as outlined at the beginning of this section. One can see the $T - \omega$ plane for

$T \geq 0, \omega \geq 0$. The black three points represent roots of the (absolute values) characteristic equation (9). The white domain represents the set (13). Hence all roots of of (9) must lie in the gray domain. Figure 1(b) shows basically the situation as outlined in the proposed stability analysis test. The white domain represents the set (16). To see that $\underline{\tau_{max}}$ defined by (17) is a lower bound of τ_{max}, observe *now* that the level sets of the function (6), which are given by the dashed-dotted lines in Figure 1(b) reveal the crucial observation that the minimum of the function (6) subject to the domain (16) is attained at (T_{max}, ω_{min}). Hence, for all roots of the characteristic equation (9), which must lie in the gray domain, τ is larger or equal to the value of characteristic equation (9) evaluated at (T_{max}, ω_{min}). Hence, (T_{max}, ω_{min}) is a lower bound of the τ_{max}. Finally, notice that it makes only sense to study the function (6) for $k = 0$ is obvious, since the objective is to find a lower bound of (T_{max}, ω_{min}). Figure 2 shows the function (6) for the domain of interest, i.e, $T \geq 0, \omega \geq 0$ and, to give a better global picture, also for negative arguments. Summarizing, the crucial observation is that the level sets of the function (6) have a particular shape (convex set). From this, the proposed stability analysis test is easy to understand and justifies graphically the proposed test. That this graphical discussion is indeed airtight, is proved in the next section.

IV. MAIN RESULTS

In this section, the main results of this paper, namely the justification of the stability analysis test in Section III and a sum of squares formulation of the stability analysis test are derived. Collecting the ideas in Section III together, the following theorem justifies the stability analysis test:

Theorem 1: Suppose that the matrix $A_0 - A_1$ has no purely imaginary eigenvalues and suppose that

$$|D(i\omega, T)| > 0 \tag{18}$$

holds for

$$\begin{aligned} &\forall T \in [0, T_{max}], \forall \omega \in [0, \infty), \\ &\forall T \in [0, \infty), \forall \omega \in [\omega_{min}, \infty). \end{aligned} \tag{19}$$

Then

$$\underline{\tau_{max}} = \frac{2}{\omega_{min}} tan^{-1}(\omega_{min} T_{max}) \tag{20}$$

is a lower bound of τ_{max}, i.e., the TDLTI system (1) under the Assumption (2) is stable for all $\tau \in [0, \underline{\tau_{max}}]$.

Proof. Due to Lemma 1, $|D(i\omega, T)| = 0$ ($D(i\omega, T) = 0$) if and only if $|\Delta(i\omega, \tau)| = 0$ ($\Delta(i\omega, \tau) = 0$), where T and τ are related via equation (6). Further, Assumption (2) guarantees that $|D(i\omega, T = 0)| > 0$ for all $\omega \in [0, \infty)$. Moreover, (18) and (19) guarantees that all roots $(\tilde{T}, \tilde{\omega})$, $\tilde{T} \geq 0$, $\tilde{\omega} \geq 0$, of D on the $i\omega$-axis must lie in domain $(T_{max}, \infty) \times [0, \omega_{min})$. The hypothesis is now that (T_{max}, ω_{min}) defines a lower bound for τ_{max} via (20). In other words,

$$\frac{2}{\tilde{\omega}} \left(tan^{-1}(\tilde{\omega}\tilde{T}) + k\frac{\pi}{2} \right) \geq \frac{2}{\omega_{min}} tan^{-1}(\omega_{min} T_{max}) \tag{21}$$

holds for all $k = 0, 1, 2, \dots$ and all $(\tilde{T}, \tilde{\omega}) \in (T_{max}, \infty) \times [0, \omega_{min})$. This hypothesis is true, if one shows, for example, that the function

$$\tau(T, \omega) = \frac{1}{\omega} tan^{-1}(\omega T), \tag{22}$$

satisfies

$$\frac{\partial}{\partial T}\tau(T, \omega) \geq 0, \quad \frac{\partial}{\partial w}\tau(T, \omega) \leq 0 \tag{23}$$

for all $T \geq 0$, $\omega \geq 0$. Namely, if $\tau = \tau(T, \omega)$ is monotonically increasing in direction T and monotonically decreasing in direction ω and since possible roots satisfy $(\tilde{T}, \tilde{\omega}) \in (T_{max}, \infty) \times [0, \omega_{min})$, (21) obviously holds, due to $\tilde{T} \geq T_{max}$ and $\tilde{\omega} \leq \omega_{min}$. Hence, it has to be shown that (23) holds. By differentiation of (22) w.r.t. to T follows immediately

$$\frac{\partial}{\partial T}\tau(T, \omega) = \frac{1}{1 + (\omega T)^2} \geq 0. \tag{24}$$

By differentiation of (22) w.r.t. ω follows

$$\frac{\partial}{\partial \omega}\tau(T, \omega) = -\frac{1}{\omega^2} tan^{-1}(\omega T) + \frac{T}{\omega(1 + (\omega T)^2)}. \tag{25}$$

Hence, it must hold

$$\frac{1}{\omega^2} tan^{-1}(\omega T) \geq \frac{T}{\omega(1 + (\omega T)^2)} \tag{26}$$

or equivalently,

$$tan^{-1}(\omega T) \geq \frac{\omega T}{1 + (\omega T)^2}. \tag{27}$$

But this is indeed true, since

$$\frac{d}{dx}a(x) \geq \frac{d}{dx}b(x) \tag{28}$$

with $a(x) = tan^{-1}(x)$ and $b(x) = \frac{x}{1+x^2}$ holds, which follows by simple algebraic manipulations and (28) implies

$$a(x) \geq b(x), \tag{29}$$

i.e., (27), due to $a(0) = b(0) = 0$. Therefore

$$\frac{\partial}{\partial \omega}\tau(T, \omega) \leq 0. \tag{30}$$

It is assumed that $det(sI - A_0 + A_1)$ has no zeros on the imaginary axis. This is because the case $T = \infty$, $\frac{1-Ti\omega}{1+Ti\omega} = -1$, in (8) is not appropriately reflected in (9), (18). In case $A_0 - A_1$ has purely imaginary eigenvalues, the stability test is still applicable. More details on this special case and how to proceed then can be found in [5], p. 36f. Alternatively, one can use instead of (5) the following substitution: $e^{-\tau s} \mapsto \frac{(1-Ts)^2}{(1+Ts)^2}$. In this case, the $2/\omega$ in relation (6) has to be replaced by $4/\omega$ and the assumption made on $A_0 - A_1$ can be skipped, cf. [24], [10]. $\square$

Theorem 1 justifies the stability analysis test proposed in Section III. In a next step, it is shown how the proposed stability analysis test can be implemented on a computer very efficiently using sum of squares techniques. Some background material on sum of squares techniques can be found in [16], [20]. Furthermore, the Matlab toolboxes [18] and [9] are available to solve sum of squares problems. Without going into details, the bottom line of sum of squares techniques is that one can check very efficiently whether a multivariable polynomial is globally positive or not by using efficient and reliable convex optimization, namely semidefinite programming algorithms. For the stability analysis test, this means one has to write down (15) together with (16) in terms of a polynomial inequality. Obviously, $|D(i\omega,T)|^2$ is a polynomial. To ensure strict positivity of $|D(i\omega,T)|^2$, the first step is to replace (15) with $|D(i\omega,T)|^2 \geq \varepsilon$, where ε is a very small constant, i.e., 10^{-4}. The second step is to incorporate the constraints (16), where the positivity test has to take place. For this, the following substitutions in (9), are used:

$$T = \frac{T_{max}}{1+u^2}, \quad \omega = \omega_{min} + v^2, \tag{31}$$

which maps from $u \mapsto T : (-\infty,\infty) \rightarrow [0, T_{max})$, $v \mapsto \omega : (-\infty,\infty) \rightarrow [\omega_{min},\infty)$ respectively. Note that the first substitution turns the characteristic equation (9) into a rational function, however, by multiplying again with $(1+u^2)^n$, one obtains again a polynomial. Hence, one arrives at the following numerical stability analysis algorithm:

Stability Analysis Algorithm using Rekasius's Substitution and Sum of Squares (RSOS): *Step 1: Choose a small* $\varepsilon > 0$*. Check if*

$$\left|(1+u^2)^n D\left(i\omega, \frac{T_{max}}{1+u^2}\right)\right|^2 \geq \varepsilon \tag{32}$$

holds for all $\omega, u \in \mathbb{R}$ *using sum of squares techniques. Check if*

$$\left|D\left(i(\omega_{min}+v^2), T^2\right)\right|^2 \geq \varepsilon \tag{33}$$

holds for all $T, v \in \mathbb{R}$ *using sum of squares techniques.*
Step 2: Compute a lower bound of τ_{max} *by*

$$\underline{\tau_{max}} = \frac{2}{\omega_{min}} tan^{-1}(\omega_{min} T_{max}). \tag{34}$$

Some additional remarks are at this place necessary. First, one would like to have the lower bound $\underline{\tau_{max}}$ as large as possible. One way to achieve is via a line search with respect to T_{max}, ω_{min}. An alternative is to maximize T_{max} and to minimize ω_{min}. This is also possible by sum of squares techniques. Second, note that (32) is satisfied for negative $\omega's$ if it is satisfied for positive $\omega's$, since the absolute value of D is symmetric with respect to ω. It is, of course, also possible to replace in (32), ω by ω^2. Notice also that the sum of squares decomposition is a sufficient condition to check positivity of multivariable polynomials. However, for certain cases which includes the case of two variables, it has been shown by Hilbert (cf. e.g. [20]), that a polynomial is nonnegative (positive) if and only if the polynomial can be written as a sum of squares of polynomials. Hence to check (32), (33) by using sum of squares techniques does not introduce any conservatism, since polynomials in the two variables u, T respectively in ω, v are investigated. In particular, in the delay-independent stability case the proposed stability test, i.e., $|D(iv^2, T^2)|^2 > 0(\varepsilon)\ \forall v, T$, is necessary and sufficient, neglecting the constant ε (cf. also [22]).

Summarizing, a new theorem for the stability analysis of TDLTI systems was proved which justifies the stability analysis test in Section III. Further, a numerical stability analysis algorithm was derived which is based on efficient sum of squares techniques. Theorem 1 is a sufficient condition and delivers a lower bound $\underline{\tau_{max}}$ for τ_{max}. However, the following examples demonstrate that this lower bound is less conservative then many stability tests based on Lyapunov-Krasovskii functionals and linear matrix inequalities.

V. EXAMPLES

Example 1. Consider the TDLTI system (1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \tag{35}$$

This is a well known benchmark example from the literature. Table I, which is taken from [4], summarizes several attempts from the literature over the past 10 years to obtain a best possible lower bound for τ_{max}. In particular, the paper from [4] comes already very close to the exact bound. However, the proposed stability analysis algorithm RSOS in Section IV obtains the best bound in comparison to all other methods in the table. Even more, the exact bound is practically achieved. The difference is caused due to ε. To check the inequalities (32) and (33), the Matlab toolbox SOSTOOLS [18] was used. The parameters are: $\varepsilon = 10^{-5}$, $T_{max} = 9.985$, and $\omega_{min} = 0.436$. T_{max} and ω_{min} were obtained via a line search.

Method	Lower bound $\underline{\tau_{max}}$
[8]	0.8571
[12]	0.99
[11]	4.3588
[3]	4.47
[7], [25], [23]	4.4721
[4]	4.4721 - 6.09
RSOS (Theorem 1)	6.1698
Exact bound τ_{max}	6.17

TABLE I
RESULTS FOR EXAMPLE 1.

Example 2. Consider the TDLTI system (1) with

$$A_0 = \begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}, \tag{36}$$

taken from [14]. By using $\varepsilon = 10^{-3}$, one obtains after a line search $T_{max} = 0.08$, $\omega_{min} = 3.2$. Hence the obtained lower bound is $\underline{\tau_{max}} = 0.1566$, which is again almost exact since $\tau_{max} = 0.1624$. Note that the coefficients of the characteristic polynomial differ by more than a factor of 10^4. Therefore, the accuracy of the semidefinite programming solver was set to 10^{-14}. An alternative is to transform the system into other coordinates.

VI. CONCLUSIONS

In this paper, an alternative delay-dependent stability analysis for TDLTI systems is proposed. In particular, a lower bound for the maximal tolerated time-delay of a TDLTI system without jeopardizing stability is derived. The theoretical contribution of this paper is an observed geometric property of the bilinear transformation which is used in Theorem 1 to establish the lower bound and consequently the numerically efficient stability test using semidefinite programming and sum of squares techniques. In comparison to Lyapunov-Krasovskii functional methods, the new approach performs well and even obtains almost the exact lower bound for a well-known benchmark example from the literature, i.e., Example 1 in Section V. Due to the use of sum of squares techniques, it is straightforward to include other analysis objectives, e.g., parameter uncertainties. This as well as other points, like the use of Theorem 1 in combination with alternative numerical methods, are not presented in the current paper because of space limitations and will be presented in a future paper. The current limitation of the new approach is that up to now only a single time-delay is considered. However, it seems that the approach can be extended to multiple delays, which is a topic of future research. For the multiple delays, one obtains polynomials in more than two variables. Although in this case the sum of squares decomposition is only a sufficient condition, these techniques are still applicable. Furthermore, considering the characteristic equation (9), or the condition (33), seems to be not the best way from a numerical point of view, since the polynomial can be ill-conditioned. Therefore, another future research topic is concerned with a better numerical formulating of the investigated problem. Summarizing, the authors believe that the new proposed time-dependent stability analysis is an interesting alternative to Lyapunov-Krasovskii functionals approaches with promising properties and which may lead to new analysis and design methods in the important research area of time-delay systems.

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