Certainty-Equivalence Feedback Design with Polynomial-Type Feedbacks which guarantee ISS

Christian Ebenbauer, Tobias Raff, and Frank Allgöwer

***Abstract*— The purpose of this note is to establish a certainty-equivalence feedback design for inverse optimally controlled affine systems. In particular, it is shown that a class of polynomial-type state feedbacks in conjunction with a globally asymptotically convergent observer leads to a globally asymptotically stable closed-loop. A key step in the proposed certainty-equivalence feedback design procedure is the identification of a new class of polynomial-type inverse optimal feedbacks which guarantees input-to-state stability with respect to measurement errors. As a consequence, the proposed certainty-equivalence feedback design has the important feature that the state feedback is allowed to contain polynomial nonlinearities of arbitrarily high degree in the unmeasured states. This feature is illustrated on an example.**

***Index Terms*— Certainty-equivalence design, separation principle, input-to-state stability, polynomial feedback, inverse optimality.**

I. Introduction

Nonlinear output feedback design is one of the most important and challenging problems in nonlinear control [10]. One reason for this is that a separated design of a global asymptotic stabilizing state feedback and a global convergent observer does not automatically lead to a global asymptotic stable closed-loop in nonlinear feedback design. Additional effort is necessary in order to guarantee global asymptotic stability, for example, either to redesign the observer or to redesign the state feedback. But often, one would like to design the state feedback and the observer completely independent from each other. However, a true modular design of the state feedback and the observer, i.e., a certainty-equivalence type implementation, is in the general nonlinear case not possible. Due to this lack, one has to assume some inherent properties in the state feedback or in the observer, in order to guarantee global asymptotic stability when the loop is going to be closed. Since in control practice, often optimal state feedbacks with respect to a certain performance measure are applied, it makes sense to assume that the state feedback satisfies a certain performance measure. Therefore, the present approach exploits the inherent robustness of optimal state feedbacks in order to establish a conceptually simple certainty-equivalence feedback design for control systems that are affine in the input. More precisely, it is shown that a class of polynomial-type inverse optimal state feedbacks in conjunction with a globally asymptotically convergent observer leads to a globally asymptotically stable closed-loop. The inverse optimality is assumed to be with respect to a classical integral performance measure of the type "u-squared plus a positive definite function of the state x". Such performance measures are well established in control theory and practice [12], [11]. Furthermore, it is assumed that the optimal feedback can be decomposed into two parts, namely a function which is globally Lipschitz plus a polynomial vector-valued function where each component depends only on one state or on a linear combination of the states. The main practical result of this note is that global asymptotic stability of the closed-loop can be established under these assumptions. Thereby, the main theoretical contribution is the established input-to-state stability (ISS) property with respect to measurement errors that can be guaranteed by the identified class of polynomial-type feedbacks. An important feature of the certainty-equivalence design is that explicit conditions are given in which the state feedback is allowed to contain polynomial nonlinearities of arbitrarily high degree in the unmeasured states. This was up to now only possible by exploiting the circle criterion [5], [3] or by exploiting structural assumptions like tringular systems, e.g., [8].

The remainder of the note is organized as follows: In Section II, the ISS property of a class of polynomial-type state feedbacks is established. The certainty-equivalence feedback design is established in Section III. In Section IV, the certainty-equivalence feedback design is illustrated on an example. Conclusions are given in Section V.

Notations. A function $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite, if $V(0) = 0, V(x) > 0, \forall x \in \mathbb{R}^n \setminus \{0\}$ and V is called a Lyapunov function candidate if it is positive definite and radially unbounded. If V is differentiable, then the row vector $V_x(x) = \nabla V(x)$ denotes the derivative of V with respect to x. $\mathcal{K}$ is the class of functions from the positive reals to the positive reals which are zero at zero, strictly increasing, and continuous. $\mathcal{K}_\infty$ is the subset in the class of $\mathcal{K}$ functions that are unbounded. Finally, the Euclidian norm of $x \in \mathbb{R}^n$ is denoted by $\|x\|$, the absolute value of $x \in \mathbb{R}$ is denoted by $|x|$ and the identity matrix is denoted by I.

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II. A Class of Polynomial-Type Feedbacks which guarantees ISS

Establishing separation principles and certainty-equivalence designs for nonlinear control systems is more involved and diverse [14], [15], [16] than for linear control systems. In the literature, there exist at least two successful concepts to establish a nonlinear separation result: The high-gain based approaches, e.g., [6], and the (i)ISS-related approaches, e.g., [2], [5], [4], [3] as well as combinations of them. In the present note, an ISS-related approach is proposed. In particular, the proposed certainty-equivalence feedback design is based upon a class of state feedbacks which guarantees a certain robustness with respect to measurement errors in order to ensure existence and boundedness of the closed-loop solutions until the observer error is sufficiently small. A well-known concept which describes this kind of robustness is ISS. Therefore, the crucial step and the main theoretical contribution in the proposed certainty-equivalence feedback design is to identify a relevant class of state feedbacks for which the ISS property with respect to measurement errors can be proven. In the following, a new class of polynomial-type state feedbacks is introduced which guarantees ISS with respect to measurement errors under the following assumptions:

Assumption 1 (System Class). The nonlinear control system is assumed to be of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^q$ is the input and $y \in \mathbb{R}^p$ the output. $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n \times q}$, and $h : \mathbb{R}^n \to \mathbb{R}^p$, are assumed to be sufficiently smooth with $f(0) = 0$, $h(0) = 0$.

Assumption 2 (State Feedback). The globally asymptotically stabilizing state feedback is assumed to be of the form

$$u = k(x) = -\frac{1}{2} R^{-1}(x) G^T(x) V_x^T(x). \tag{2}$$

That is, it is assumed that the state feedback (2) is (inverse) optimal with respect to the performance measure

$$V(x(0)) = \int_0^\infty q(x(t)) + u^T(t) R(x(t)) u(t)\, dt, \tag{3}$$

i.e., the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} V_x(x) f(x) + V_x(x) G(x) k(x) \\ + q(x) + k^T(x) R(x) k(x) = 0, \end{aligned} \tag{4}$$

is satisfied, where $q(x) \geq c\|x\|^2$, $c > 0$ and R is a positive definite matrix function with $\lambda_{min} I \leq R(x) \leq \lambda_{max} I$, $\lambda_{max} > \lambda_{min} > 0$. Furthermore, V is assumed to be a positive definite, radially unbounded C^1 function.

Then the following statement can be made.

Theorem 1: Suppose that Assumption 1 and 2 hold. Moreover, suppose that the state feedback (2) is of the form

$$u = k(x) = m(x) + p(x_i), \tag{5}$$

where $m : \mathbb{R}^n \to \mathbb{R}^q$, $m(0) = 0$, is globally Lipschitz, i.e., $\|m(x+e) - m(x)\| \leq \gamma \|e\|$, with a Lipschitz constant γ and $p : \mathbb{R}^n \to \mathbb{R}^q$ is of the form

$$p(x_i) = \begin{bmatrix} p_1(x_{i_1}) \\ \vdots \\ p_q(x_{i_q}) \end{bmatrix}, \tag{6}$$

where the components p_j are polynomial functions in a single state x_{i_j}, $i_j \in \{1...n\}$, $j = 1...q$. Then the control system

$$\dot{x} = f(x) + G(x) k(x+e) \tag{7}$$

is input-to-state stable with respect to the measurement error $e \in \mathbb{R}^n$.

Notice that the notation $p(x_i)$ means that p is formed by univariate polynomials and each polynomial depends on a certain state variable, not necessarily different from each other, e.g. p might be of the form $[p_1(x_8)\ p_2(x_1)\ p_3(x_3)\ p_4(x_8)]^T$ ($n = 8$, $q = 4$).

Proof. By using the value function V in (3) as an ISS Lyapunov function candidate, it has to be shown that there exist a class $\mathcal{K}$ function ρ and a class $\mathcal{K}_\infty$ function β such that

$$V_x(x)[f(x) + G(x) k(x+e)] \leq -\beta(\|x\|), \tag{8}$$

holds, whenever $\|x\| \geq \rho(\|e\|)$ [13], [9]. In other words, it has to be shown that

$$\begin{aligned} V_x(x) f(x) \\ + V_x(x) G(x)[m(x+e) + p(x_i + e_i)] < -\beta(\|x\|) \end{aligned} \tag{9}$$

holds for any given (fixed) e whenever $\|x\|$ is sufficiently large. The proof is divided into two parts.

Part 1. In the first part of the proof, an upper estimate of the left-hand side of (9) is derived, by successively overestimating the left-hand side. First, add and subtract $V_x(x)G(x)[m(x) + p(x_i)]$ from the left-hand side of (9), which leads to

$$\begin{aligned} V_x(x)[f(x) + G(x)(m(x) + p(x_i))] \\ + V_x(x) G(x)[m(x+e) - m(x) + p(x_i + e_i) - p(x_i)]. \end{aligned} \tag{10}$$

Using the HJB inequality (4) and the relation (2), one obtains

$$\begin{aligned} -q(x) - [m(x) + p(x_i)]^T R(x) [m(x) + p(x_i)] \\ -2[m(x) + p(x_i)]^T R(x) [\Delta m(x,e) + \Delta p(x_i, e_i)] \end{aligned} \tag{11}$$

with $\Delta m(x,e) = m(x+e) - m(x)$ and $\Delta p(x_i, e_i) = p(x_i + e_i) - p(x_i)$. The expression (11) is bounded from above by

$$\begin{aligned} -q(x) - \lambda_{min} \|m(x) + p(x_i)\|^2 \\ -2[m(x) + p(x_i)]^T R(x) [\Delta m(x,e) + \Delta p(x_i, e_i)] \end{aligned} \tag{12}$$

due to $\lambda_{min} I \leq R(x) \leq \lambda_{max} I$. Using Young's inequality, i.e., $2a^T b \leq \alpha_1^{-1} \|a\|^2 + \alpha_1 \|b\|^2$, $a, b \in \mathbb{R}^n$, $\alpha_1 > 0$, with

$a = R(x)[m(x)+p(x_i)]$ and $b = \Delta m(x,e) + \Delta p(x_i, e_i)$, one obtains

$$\begin{aligned} -q(x) - \lambda_{min}\|m(x)+p(x_i)\|^2 \\ +\frac{\lambda_{max}^2}{\alpha_1}\|m(x)+p(x_i)\|^2 \\ +\alpha_1\|\Delta m(x,e) + \Delta p(x_i,e_i)\|^2 \end{aligned} \tag{13}$$

as an upper bound for (12). Choosing α_1 such that $\frac{\lambda_{min}}{2} = \frac{\lambda_{max}^2}{\alpha_1}$ and due to $q(x) \geq c\|x\|^2$,

$$\begin{aligned} -c\|x\|^2 \\ -\frac{\lambda_{min}}{2}\left[\|m(x)\|^2 + 2m^T(x)p(x_i) + \|p(x_i)\|^2\right] \\ +\frac{2\lambda_{max}^2}{\lambda_{min}}\|\Delta m(x,e) + \Delta p(x_i,e_i)\|^2 \end{aligned} \tag{14}$$

is an upper bound for (13). Applying Young's inequality $2a^Tb \leq \alpha_2^{-1}\|a\|^2 + \alpha_2\|b\|^2$ to the expression $2m(x)^Tp(x_i)$ yields to

$$\begin{aligned} -c\|x\|^2 \\ -\frac{\lambda_{min}}{2}\left[(1-\alpha_2^{-1})\|m(x)\|^2 + (1-\alpha_2)\|p(x_i)\|^2\right] \\ +\frac{2\lambda_{max}^2}{\lambda_{min}}\|\Delta m(x,e) + \Delta p(x_i,e_i)\|^2, \end{aligned} \tag{15}$$

which is an upper bound of (14). Next notice that one can choose $\alpha_2 < 1$ but such that the linear-like part of (15) is still negative definite, i.e.,

$$-c\|x\|^2 + \frac{\lambda_{min}}{2}\underbrace{(\alpha_2^{-1}-1)}_{>0}\|m(x)\|^2 < 0 \tag{16}$$

with α_2 such $\frac{\lambda_{min}}{2}(\alpha_2^{-1}-1)\gamma^2 < c$, due to the Lipschitz property $\|m(x)\| \leq \gamma\|x\|$. Therefore,

$$\begin{aligned} -\tilde{c}\|x\|^2 - \tilde{d}\|p(x_i)\|^2 \\ +\frac{2\lambda_{max}^2}{\lambda_{min}}\|\Delta m(x,e) + \Delta p(x_i,e_i)\|^2 \end{aligned} \tag{17}$$

with $\tilde{c} = c - \frac{\lambda_{min}}{2}(\alpha_2^{-1}-1)\gamma^2 > 0$ and $\tilde{d} = (1-\alpha_2) > 0$, is an upper bound for (15). Finally, by using the triangular inequality and $(\|a\|+\|b\|)^2 \leq 2(\|a\|^2+\|b\|^2)$ as well as the Lipschitz property $\|\Delta m(e,x)\| \leq \gamma\|e\|$,

$$\begin{aligned} -\tilde{c}\|x\|^2 - \tilde{d}\|p(x_i)\|^2 \\ +\frac{4\lambda_{max}^2}{\lambda_{min}}(\gamma^2\|e\|^2 + \|\Delta p(x_i,e_i)\|^2) \end{aligned} \tag{18}$$

is an upper bound for (17), and by construction, an upper bound for the left-hand side of (9).

Part 2. In the second part of the proof, it is shown that (18) satisfies

$$\begin{aligned} -\tilde{c}\|x\|^2 - \tilde{d}\|p(x_i)\|^2 \\ +\frac{4\lambda_{max}^2}{\lambda_{min}}(\gamma^2\|e\|^2 + \|\Delta p(x_i,e_i)\|^2) < -\beta(\|x\|) \end{aligned} \tag{19}$$

for a sufficiently large $\|x\|$ with $\beta(\|x\|) = \frac{\tilde{c}}{2}\|x\|^2$, which implies the desired property (8). Thus, it has to be shown that

$$\begin{aligned} -\frac{\tilde{c}}{2}\|x\|^2 + \tilde{\lambda}\gamma^2\|e\|^2 \\ -\tilde{d}\sum_{j=1}^{q} p_j^2(x_{i_j}) + \tilde{\lambda}\sum_{j=1}^{q}[p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})]^2 < 0 \end{aligned} \tag{20}$$

holds for sufficiently large $\|x\|$ with $\tilde{\lambda} = \frac{4\lambda_{max}^2}{\lambda_{min}}$. Let's assume that p_j is of degree d_j. Then

$$p_j^2(x_{i_j}) = \sum_{k=0}^{2d_j} a_k x_{i_j}^k \tag{21}$$

is a positive semidefinite polynomial, in particular $a_{2d_j} > 0$, which depends on a single state x_{i_j}. Furthermore,

$$[p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})]^2 = \sum_{k=0}^{2d_j-2} b_k(e)x_{i_j}^k \tag{22}$$

with b_k polynomial, is a positive semidefinite polynomial of degree $2d_j - 2$ in x_{i_j}, since for a univariate polynomial $p(s)$ of degree d it holds that $p(s) - p(s+t)$, with a fixed t, is again a polynomial but of degree $d-1$. The sum in (22) can be bounded from above by

$$\begin{aligned} [p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})]^2 \\ \leq \sum_{k=0}^{2d_j-2} \frac{k}{k+1}|x_{i_j}|^{k+1} + \frac{1}{k+1}|b_k(e)|^{k+1} \end{aligned} \tag{23}$$

using the inequality $y \cdot x^k \leq \frac{|x^k|^a}{a} + \frac{|y|^b}{b}$, $\frac{1}{a}+\frac{1}{b} = 1$, where a depends on k via $a = 1 + \frac{1}{k}$. Therefore, the expression

$$-\tilde{d}p_j^2(x_{i_j}) + \tilde{\lambda}[p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})]^2 \tag{24}$$

in the sum (20) can be bounded from above by

$$\begin{aligned} -\tilde{d}a_{2d_j}x_{i_j}^{2d_j} + \tilde{\lambda}\frac{2d_j-2}{2d_j-1}|x_{i_j}|^{2d_j-1} \\ -\tilde{d}a_{2d_j-1}x_{i_j}^{2d_j-1} + \tilde{\lambda}\frac{2d_j-3}{2d_j-2}|x_{i_j}|^{2d_j-2}\ldots \\ -\tilde{d}a_0 + \tilde{\lambda}|b_0(e)| + \ldots + \frac{\tilde{\lambda}}{2d_j-1}|b_{2d_j-2}(e)|^2, \end{aligned} \tag{25}$$

where the terms are sorted by the exponents of x_{i_j}. Since, $\tilde{d}a_{2d_j} > 0$, (25) becomes negative whenever $|x_{i_j}|$ becomes sufficiently large. This implies that

$$-\tilde{d}p_j^2(x_{i_j}) + \tilde{\lambda}[p_j(x_{i_j}+e_{i_j}) - p_j(x_{i_j})]^2 < 0 \tag{26}$$

holds whenever $|x_{i_j}|$ is sufficiently large. Hence, whenever the absolute values of the states in $\{x_{i_1}...x_{i_q}\}$ become sufficiently large, then inequality (20) holds. Finally, in the case if $\|x\|$ becomes sufficiently large, but not any of the states in $\{x_{i_1}...x_{i_q}\}$, then still the term $\frac{\tilde{c}}{2}\|x\|^2$ becomes sufficiently large, such that for a given (fixed) e, the inequality (20) holds. □

The next corollary is a generalization of Theorem 1, which enables that the components of the polynomial nonlinearities may depend on a linear combination of the states and not necessarily on a single state.

Corollary 1: Suppose that Assumption 1 and 2 hold. Moreover, suppose that the state feedback (2) is of the form

$$u = k(x) = m(x) + p(c_i^T x), \tag{27}$$

where $m : \mathbb{R}^n \to \mathbb{R}^q$, $m(0) = 0$, is globally Lipschitz, i.e., $\|m(x+e) - m(x)\| \leq \gamma\|e\|$, with a Lipschitz constant γ and $p : \mathbb{R}^n \to \mathbb{R}^q$ is of the form

$$p(c_i^T x) = \begin{bmatrix} p_1(c_{i_1}^T x) \\ \vdots \\ p_q(c_{i_q}^T x) \end{bmatrix}, \tag{28}$$

where the components p_j are polynomial functions and $c_{i_j} \in \mathbb{R}^n$, $i_j \in \{1...n\}$, $j = 1...q$. Then the control system

$$\dot{x} = f(x) + G(x)k(x+e) \tag{29}$$

is input-to-state stable with respect to the measurement error $e \in \mathbb{R}^n$.

Proof. The idea of the proof is to transform the feedback (27) in the form (5) using a linear transformation $\Phi\xi = x$, with a nonsingular matrix Φ such that $\xi_j = c_{i_j}^T x$, $j = 1...q$. Then the transformed state feedback is

$$u = \tilde{k}(\xi) = k(\Phi\xi) = m(\Phi\xi) + p(\xi_i), \tag{30}$$

which is of the form (5). Notice that $m(\Phi\xi)$ is globally Lipschitz since it is a linear transformation. The rest of the proof shows that the transformed control system (1), i.e.,

$$\begin{aligned} \dot{\xi} &= \tilde{f}(\xi) + \tilde{G}(\xi)u \\ y &= h(\Phi\xi), \end{aligned} \tag{31}$$

with $\tilde{f}(\xi) = \Phi^{-1}f(\Phi\xi)$ and $\tilde{G}(\xi) = \Phi^{-1}G(\Phi\xi)$ satisfies a HJB equation of the form (4). In particular, from the HJB equation (4), it follows that

$$\begin{aligned} &V_x(\Phi\xi)[f(\Phi\xi) + G(\Phi\xi)k(\Phi\xi)] \\ &+q(\Phi\xi) + k(\Phi\xi)^T R(\Phi\xi)k(\Phi\xi) = 0, \end{aligned} \tag{32}$$

or equivalently

$$\begin{aligned} &V_x(\Phi\xi)\Phi[\tilde{f}(\xi) + \tilde{G}(\xi)k(\Phi\xi)] \\ &+q(\Phi\xi) + k(\Phi\xi)^T R(\Phi\xi)k(\Phi\xi) = 0. \end{aligned} \tag{33}$$

Define now a new value function $\tilde{V}(\xi) = V(\Phi\xi)$, then the gradient $\tilde{V}_\xi(\xi) = V_x(\Phi\xi)\Phi$. Hence, (33) turns into

$$\tilde{V}_\xi(\xi)[\tilde{f}(\xi) + \tilde{G}(\xi)\tilde{k}(\xi)] + \tilde{q}(\xi) + \tilde{k}(\xi)^T \tilde{R}(\xi)\tilde{k}(\xi) = 0. \tag{34}$$

which is a HJB equation of the form (4), with $\tilde{q}(\xi) = q(\Phi\xi)$, $\tilde{R}(\xi) = R(\Phi\xi)$. Finally, by using Theorem 1, Corollary 1 follows. □

Remark 1: The proof of Corollary 1 can be generalized by replacing the linear transformation $x = \Phi\xi$ with a Lipschitz transformation $x = \Phi(\xi)$, where Φ is globally Lipschitz and sufficiently smooth. Therefore, Theorem 1 can be further extended such that the polynomial nonlinearity (28) can be replaced by:

$$p(c_i(x)) = \begin{bmatrix} p_1(c_{i_1}(x)) \\ \vdots \\ p_q(c_{i_q}(x)) \end{bmatrix}, \tag{35}$$

where c_{i_j} are smooth nonlinearities which are globally Lipschitz.

Summarizing, a class of polynomial-type feedbacks has been derived in Theorem 1 and Corollary 1 which guarantees the ISS property with respect to measurement errors.

III. Certainty-Equivalence Feedback Design

In the following, it is shown how the identified class of polynomial-type state feedbacks can be applied to design certainty-equivalence feedbacks.

Assumption 3 (State Observer). It is assumed that a state observer for the estimated state $\hat{x}$ for the control system (1) with a globally uniform asymptotic observer error dynamics

$$\dot{e} = a(e, x), \tag{36}$$

$e = x - \hat{x}$, $e \in \mathbb{R}^n$ is known. More precisely, it is assumed that there exists a Lyapunov function W such that

$$W_e(e)a(e, x) < -\alpha(W(e)) \tag{37}$$

for all nonzero e, x, where α is a positive define function.

Then, under this assumption, the question arise whether the closed-loop given by the following certainty-equivalence type implementation

$$\begin{aligned} \dot{x} &= f(x) + G(x)k(x+e) \\ \dot{e} &= a(e, x), \end{aligned} \tag{38}$$

and defined by the Assumptions 1-3, is globally asymptotically stable (see Fig. 1)?

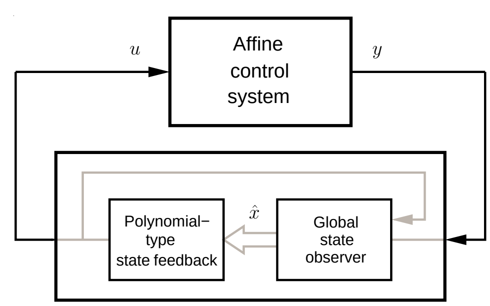

Fig. 1. Closed-loop system.

By inspection of the closed-loop system (38) together with the convergence property (37) of the observer and the results obtained in the previous section, the following certainty-equivalence feedback design with guaranteed global asymptotic stability follows.

Theorem 2: Suppose that Assumption 1-3 hold. Moreover, suppose that the state feedback (2) is of the form (27), (28). Then the closed-loop (38) is globally asymptotically stable.

Proof. Consider the closed-loop (38), i.e.,

$$\dot{x} = f(x) + G(x)k(x+e) \quad (39)$$

$$\dot{e} = a(e,x). \quad (40)$$

By Corollary 1, the x-subsystem (39) is input-to-state stable with respect to e. Moreover, the e-subsystem (40) satisfies the uniform convergence property (37), i.e., the right-hand side of (37) does not depend on x. Thus (39), (40) together with (37) has a cascade-like behavior, i.e., e converges to zero and the ISS property guarantees that the state x stays bounded while e goes to zero. Such situations are well investigated in the literature and one can employ for example ISS theory, cf. e.g. Lemma 4.7 [9], to show that (39), (40) is globally asymptotically stable. □

Remark 2: It is possible to extend the results in such a way that global asymptotic stability is guaranteed, if one adjoins a term in the state feedback which depends only on the output measurements, i.e., $u = m(x) + p(x_{i_j}) + r(y)$.

Summarizing, a certainty-equivalence feedback design for inverse optimally controlled affine systems is established under the assumption that the optimal state feedback can be transformed into a form such that it can be written as a globally Lipschitz function plus a polynomial-type nonlinearity where each component depends on a single state. Even so this might sound restrictive, it includes obviously all state feedbacks which are globally Lipschitz. Furthermore, the certainty-equivalence design allows to use polynomial nonlinearities of arbitrarily high degree in unmeasured states for global output feedback design, which was up to now only possible by exploiting the circle criterion [5], [3] or by exploiting structural assumptions like triangular systems, e.g., [8]. The main ingredient is the established class of polynomial-type feedbacks which guarantees ISS based on the inherent robustness of the HJB for affine control systems and the growth rate of polynomial functions.

IV. EXAMPLE

In the following, an example illustrates the results established in the Section III. Consider the affine control system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 4x_2^3 \\ x_2^3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \quad (41)$$

Note that the control system is unstable. The stabilizing state feedback

$$u = k(x) = -x_1 - x_2 - x_2^3, \quad (42)$$

which is polynomial in the unmeasured state x_2, is of form (5) and is optimal with respect to

$$V(x(0)) = \int_0^\infty q(x(t)) + \frac{1}{4}u^2(t)\,dt, \quad (43)$$

where $q(x) \geq \frac{1}{2}\|x\|^2$, i.e.,

$$q(x) = 3x_1^2 + 2x_1x_2 + x_2^2 + x_2^4 \quad (44)$$

and the value function V of the HJB equation (4) is

$$V(x) = \frac{1}{4}(2x_1^2 + 2x_2^2 + x_2^4). \quad (45)$$

Since the state feedback (42) satisfies all conditions of Theorem 2 and since a globally uniform convergence observer can be constructed using the observer proposed in [5], a certainty equivalence design is feasible.

V. CONCLUSIONS

In this note, a new class of polynomial-type inverse optimal state feedbacks has been identified which is applicable to certainty-equivalence feedback design. It has been proved that this class of polynomial-type feedback guarantees ISS with respect to measurements errors. Moreover, based on this class of polynomial-type feedbacks, a certainty-equivalence feedback design was proposed which guarantees a globally asymptotically stable closed-loop for an uniform converging observer error dynamics. The main assumptions in this note are that the state feedback is inverse optimal with respect to a classical integral performance measure and that the optimal state feedback can be decomposed or transformed into two parts, namely a function which is globally Lipschitz plus a polynomial vector-valued function where each component depends on a single state. Therefore, a conceptually simple certainty-equivalence feedback design for inverse optimally controlled affine systems is established under rather easily verifiable conditions. Moreover, since the introduced class of polynomial-type inverse optimal state feedbacks guarantees ISS for any type of state disturbance, this class of feedbacks could be also interesting for other applications beside output feedback. The result established here is in particular interesting for polynomial control systems, since for this class of control systems, systematic methods exist to design globally stabilizing state feedbacks and observers, e.g., cf. [7] and references therein. Another important feature of the result is that, for a practical relevant problem setup, explicit conditions are given in which the state feedback is allowed to contain polynomial nonlinearities of arbitrarily high degree in unmeasured states. Hence, new conditions for the important problem of certainty-equivalence feedback design are derived in the present note. Interesting for future research is the question when does an optimal state feedback design yields a feedback of the form exploited in this note. This is especially of interest for control strategies which do not explicitly deliver a feedback in closed form, e.g., nonlinear model predictive control [1].

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