Constrained Extremum Seeking: a Modified-Barrier Function Approach ⋆

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Abstract: In this paper, we address the problem of steering the input of a convex function to a value that minimizes the function under a convex constraint. We consider the case where the constraint cannot be violated of more than a user-defined value during the whole transient phase. The mathematical expression of both the cost function and the constraint are assumed to be unknown. The only information available are the on-line values of the cost and the constraint. To tackle this problem, an optimization law, based on a modified-barrier function, and involving the gradient of both the cost function and the constraint, is firstly designed. The Lie bracket formalism is then exploited to approximate this law, by combining time-periodic signals with the on-line measurements of both the cost and the constraint. The stability property of the resulting constrained extremum seeking system is proved, and its effectiveness is shown in simulation.

Keywords: Constrained Optimization, Extremum Seeking, Singular Perturbations, Data-based Control, Modified Barrier-Function

1. INTRODUCTION

In practical applications, the aim is often to optimize a process variable, the *cost*, while enforcing some safety margins, formulated as *constraints*. One may think to the case of battery charging, where one wants to minimize the charging time, while ensuring safety features, formulated notably as constraints on the maximal current and temperature (see e.g. Liu et al. (2017) or Zhang et al. (2017)). Another example is the production of chemicals in a reactor. In that case, one aims to maximize the production, while enforcing some constraints, such as the maximal temperature or the concentration of undesirable products (see e.g. Simon et al. (2008) or Pahija et al. (2013)).

In several cases, the relations between the control inputs and both the cost and the constraints are only partially known. Model-free extremum seeking is a class of real-time optimization methods suited for those cases. Typically, extremum seeking systems steer the control inputs towards the cost optimizer, by combining time-periodic signals with the on-line measurement of the cost (see e.g. Ariyur and Krstic (2003), Suttner and Dashkovskiy (2017) or Labar et al. (2018b)). Several extremum seeking systems considering constrained optimization problems have been proposed in the literature. One may for instance mention the schemes based on Lagrangian functions, presented in Dürr et al. (2013a) and Grushkovskaya and Ebenbauer (2016), and the ones based on penaly/barrier functions given in DeHaan and Guay (2005) and Guay et al. (2015). The schemes based on the combination of penalty/barrier functions may enforce the satisfaction of the constraints at all time. However, this requires an adequate tuning of the parameters (e.g. the value of the transition between the barrier and penalty functions). Extremum seeking schemes based on Lagrangian are, for instance, well suited for distributed optimization problems. However, the constraints may be (severely) violated during the transient, which can be problematic in practical applications.

In this paper, we aim at designing a constrained extremum seeking scheme ensuring that, at any time, and independently of the tuning parameters, the constraints are not violated of more than a user-defined value. To do so, we will combine a so-called *modified barrier function* with a saddle point dynamics. Note that allowing a user-defined (slight) violation of the constraints during the transient is something typical in many practical applications. One may think to the maximal current intensity inside a wire, the maximal torque delivered by a DC motor, or the maximal temperature in a chemical reactor.

To the best of our knowledge, the combination of a modified barrier function with saddle point dynamics has never been studied in details in the framework of extremum seeking. Although a similar combination has been proposed in Dürr and Ebenbauer (2012), the gradient of the cost was assumed to be known, and no theoretical analysis was carried out.

The remainder of the paper is organized as follows. The notations and definitions, together with the Lie bracket approximation approach, are introduced in Section 2.

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The considered problem is formally stated in Section 3. Section 4 presents the proposed approach together with its stability property. Finally, simulation results are presented in Section 5.

2. PRELIMINARIES

2.1 Notations and Definitions

The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $||x||$. $\mathbb{R}_{>0}$ and $\mathbb{Q}_{>0}$ are the sets of strictly positive real numbers and strictly positive rational numbers, respectively. $\mathbf{LCM}(k_1, k_2, ..., k_n)$ stands for the Least Common Multiple of $\{k_1, k_2, ..., k_n\}$. ∇h represents the gradient of the differentiable function $h : \mathbb{R}^n \to \mathbb{R}$. The Jacobian of a differentiable map $f : \mathbb{R}^n \to \mathbb{R}^m$ is denoted by $\frac{\partial f}{\partial x}$. The Lie bracket of two differentiable vector fields $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$, denoted by $[f, g](x)$, is defined by $\frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$. A $\delta-$neighbourhood of a set $\mathcal{S} \subset \mathbb{R}^n$, with $\delta \in \mathbb{R}_{>0}$, with respect to a set $\mathcal{R} \subseteq \mathbb{R}^n$, is denoted by $U_{\mathcal{R}}^{\mathcal{S}}(\delta)$, and defined by $U_{\mathcal{R}}^{\mathcal{S}}(\delta) := \{x \in \mathcal{R} : \inf_{y \in \mathcal{S}} ||x - y|| < \delta\}$. The closure of an open set $\mathcal{R} \subset \mathbb{R}^n$ is denoted by $\overline{\mathcal{R}}$.

Furthermore, in the sequel, we will refer to the following definitions:

Definition 1. Let $\Omega \subseteq \mathbb{R}^{n_x}$. A point $(x^*, \lambda^*) \in \Omega \times \mathbb{R}^{n_\lambda}_{\geq 0}$ is a saddle point for the function $L(x, \lambda) : \Omega \times \mathbb{R}^{n_\lambda}_{\geq 0} \to \mathbb{R}$ if

$$L(x^*, \lambda) \leq L(x^*, \lambda^*) \leq L(x, \lambda^*), \forall (x, \lambda) \in \Omega \times \mathbb{R}^{n_\lambda}_{\geq 0}. \quad (1)$$

Definition 2. Consider the constrained optimization problem $\min_{x \in \mathbb{R}^{n_x}} h(x)$ such that $g(x) \leq 0$, where $h(x) : \mathbb{R}^{n_x} \to \mathbb{R}$ and $g(x) : \mathbb{R}^{n_x} \to \mathbb{R}$ are convex functions of class C^1. Let $L(x, \lambda) := h(x) + \lambda g(x)$ be the associated Lagrangian function. A point $(x^*, \lambda^*) \in \mathbb{R}^{n_x} \times \mathbb{R}_{\geq 0}$ satisfies the Karush-Kuhn-Tucker (KKT) conditions if:

(1) $g(x^*) \leq 0$
(2) $\lambda^* \geq 0$
(3) $\lambda^* g(x^*) = 0$
(4) $\nabla_x L(x^*, \lambda^*) = 0$.

Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact set and $\mathcal{R} \subseteq \mathbb{R}^n$ be an open set such that $\overline{\mathcal{R}} \cap \mathcal{S}$ is non-empty. Similarly to Dürr et al. (2013b), we define (semi)-regional (practical) uniform asymptotic stability of $\mathcal{S}$ in $\mathcal{R}$ as follows:

Definition 3. The set of points $\mathcal{S}$ is Regionally Uniformly Asymptotically Stable (RUAS) in $\mathcal{R}$ for the n-dimensional system $\dot{x} = f(t, x)$ if, for every $\delta_B \in \mathbb{R}_{>0}$ and $\delta_V \in \mathbb{R}_{>0}$, there exist $\delta_Q \in \mathbb{R}_{>0}$ and $\delta_W \in \mathbb{R}_{>0}$ such that, for every $t_0 \in \mathbb{R}$, the following hold:

(1) Boundedness: $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V) \Rightarrow x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_W)$, $\forall t \geq t_0$;
(2) Stability: $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q) \Rightarrow x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, $\forall t \geq t_0$;
(3) Asymptotic Convergence:
$x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V) \Rightarrow \limsup_{t \to \infty} \inf_{y \in \mathcal{S}} ||x(t) - y|| = 0$.

Definition 4. The set of points $\mathcal{S}$ is semi-Regionally Practically Uniformly Asymptotically Stable (sRPUAS) in $\mathcal{R}$ for the n-dimensional system $\dot{x} = f(t, x, \epsilon)$, with the vector of parameters $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_{n_\epsilon}]^T$, if the following holds. For every $\delta_B \in \mathbb{R}_{>0}$ and $\delta_V \in \mathbb{R}_{>0}$, there exist a $\delta_Q \in \mathbb{R}_{>0}$, a $\delta_W \in \mathbb{R}_{>0}$, and an $\epsilon_1^* \in \mathbb{R}_{>0}$ such that, for every $\epsilon_1 \in (0, \epsilon_1^*)$, there exists an $\epsilon_2^* \in \mathbb{R}_{>0}$ such that, for every $\epsilon_2 \in (0, \epsilon_2^*)$,..., there exists an $\epsilon_{n_\epsilon}^* \in \mathbb{R}_{>0}$ such that, for every $\epsilon_{n_\epsilon} \in (0, \epsilon_{n_\epsilon}^*)$, there exists a $t_1 \in \mathbb{R}$ such that, for every $t_0 \in \mathbb{R}$, the following three properties are fulfilled:

(1) Boundedness: $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V) \Rightarrow x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_W)$, $\forall t \geq t_0$;
(2) Stability: $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q) \Rightarrow x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, $\forall t \geq t_0$;
(3) Practical Convergence:
$x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V) \Rightarrow x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, $\forall t \geq t_1 + t_0$.

2.2 First Order Lie Bracket Approximation Approach

In this section, we remind the notion of an associated Lie bracket system, together with some of its properties. Those properties will be exploited to design and analyze the proposed extremum seeking scheme.

Consider an input-affine system

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{l} f_i(x(t)) \sqrt{\omega} u_i(k_i \omega t), \quad (2)$$

with $x(t) \in \mathbb{R}^n$ the state vector of the system, $u_i(t) \in \mathbb{R}$ the control inputs, $\omega \in \mathbb{R}_{>0}$ and $k_i \in \mathbb{Q}_{>0}$.

Let system (2) satisfy the following Assumption:

Assumption 5. (Conditions on u_i) For all $i \in \{1, ..., l\}$:

(1) $u_i(t) : \mathbb{R} \to \mathbb{R}$ is a measurable bounded function.
(2) $u_i(t)$ is 2π-periodic, i.e. $u_i(t) = u_i(t + 2\pi), \forall t \in \mathbb{R}$;
(3) $u_i(t)$ has zero mean on one period, i.e. $\int_0^{2\pi} u_i(t) dt = 0$.

Furthermore, suppose that the vector fields $f_i(x)$ in (2) are differentiable. Then, we can associate a so-called *Lie bracket system* with (2), namely

$$\dot{\overline{x}} = f_0(\overline{x}) + \sum_{\substack{1 \leq i < l \\ i < j \leq l}} [f_i, f_j](\overline{x}) \gamma_{ij}, \quad (3)$$

where we introduced

$$\gamma_{ij} := \frac{\omega}{T} \int_0^T \int_0^s u_j(k_j \omega s) u_i(k_i \omega p) \, dp \, ds, \quad (4)$$

with $T = \frac{2\pi}{\omega} \mathbf{LCM}(k_1^{-1}, k_2^{-1}, ..., k_l^{-1})$.

The distance between the trajectory of the input-affine system (2) and the trajectory of the associated Lie bracket system (3) can be characterized by the following Lemma:

Lemma 6. Let $\mathcal{R} \subseteq \mathbb{R}^n$ be an open convex set, and suppose that the vector fields $f_i(x) : \mathcal{R} \to \mathbb{R}^n$ are of class C^2. Furthermore, assume that, $\forall i, j \in \{0, 1, ..., l\}$, the vector fields $f_i(x)$ and the Jacobian matrices $\frac{\partial f_i(x)}{\partial x}$ and $\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right)$ are bounded on every open bounded set $\mathcal{A} \subseteq \mathcal{R}$. Let $t_f \in \mathbb{R}_{>0}$ and the bounded set $\mathcal{V} \subseteq \mathcal{R}$ be such that the trajectory of system (3), with initial condition $\overline{x}(0) \in \mathcal{V}$, is unique and absolutely continuous for $t \in [0, t_f]$. Furthermore, assume that the set $\mathcal{R}$ is positively invariant for system (2). Then, under Assumption 5, for every $D \in \mathbb{R}_{>0}$, there exists an $\omega^* \in \mathbb{R}_{>0}$ such that, for all $\omega \in (\omega^*, \infty)$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathcal{V}$, the trajectories of systems (2) and (3), starting at $\overline{x}_0 = x_0$ satisfy

$$||x(t) - \overline{x}(t)|| < D, \forall t \in [t_0, t_0 + t_f]. \quad (5)$$

Proof. The proof is reported in Appendix A.

A direct consequence of Lemma 6 is that the stability property of the input-affine system (2) can be deduced from the stability property of its associated Lie bracket system (3). Such a result is stated in the following Lemma:

Lemma 7. Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact set and $\mathcal{R} \subseteq \mathbb{R}^n$ be an open convex set such that $\mathcal{S} \cap \overline{\mathcal{R}}$ is non-empty. Suppose that the vector fields $f_i(x) : \mathcal{R} \to \mathbb{R}^n$ are of class C^2 and that, $\forall i, j \in \{0, 1, ..., l\}$, the vector fields $f_i(x)$, and the Jacobian matrices $\frac{\partial f_i(x)}{\partial x}$ and $\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right)$ are bounded on every open bounded set $\mathcal{A} \subseteq \mathcal{R}$. Furthermore, let the set $\mathcal{S}$ be regionally uniformly asymptotically stable in $\mathcal{R}$ for system (3) and let system (2) be positively invariant in $\mathcal{R}$. Then, under Assumption 5, the set $\mathcal{S}$ is semi-regionally practically uniformly asymptotically stable in $\mathcal{R}$ for system (2), with the parameter ω^{-1}.

Proof. The proof is given in Appendix B. ■

Remark 8. Lemma 7 differs from Lemma 1 of Dürr et al. (2013b), since the vector fields are only defined in a region of $\mathbb{R}^n$. Furthermore, it is based on a slightly different definition of stability. The need of such extension is demonstrated in the sequel.

3. PROBLEM STATEMENT

In this section, the problem addressed in this paper is formally stated.

Consider the following constrained optimization problem

$$\min_{x\in\mathbb{R}^n} h(x) \quad \text{s.t.} \quad g(x) \leq 0, \tag{6}$$

where the functions $h(x)$ and $g(x)$ satisfy the following Assumption:

Assumption 9. The cost function $h(x)$ is strictly convex and belong to class C^2. Furthermore, the constraint $g(x)$ is a convex function of class C^2.

Remark 10. Assuming strict convexity of the cost function is quite standard in constrained extremum seeking in order to obtain non-local stability results (see e.g. Dürr et al. (2013b), Grushkovskaya and Ebenbauer (2016) or Guay et al. (2015)), as we aim for in this work.

In the framework of this paper, we consider that x is governed by the control system $\dot{x} = u$, and we assume that the cost function $h(x)$ and the constraint $g(x)$ are mathematically unknown. The only information available is the value of $h(x(t))$ and $g(x(t))$, at any time t.

Our aim is to steer $x(t)$ towards the solution of (6), while ensuring that

$$g(x(t)) \leq \beta^{-1}, \forall t \geq t_0, \tag{7}$$

where $\beta \in \mathbb{R}_{>0}$ is a user-defined value.

4. MAIN RESULTS

To address the problem, two main steps will be followed. Firstly, we will assume that the gradients of both the cost function and the constraint are available. Based on a modified barrier function and a saddle point dynamics, a control law will be designed to solve (6)-(7). Secondly, the Lie bracket formalism (Section 2.2) will be used to approximate this control law, by only using the on-line measurements of the cost and the constraint.

4.1 Saddle Point Dynamics with Modified Barrier Functions

To tackle the problem (6)-(7), we will make use of the so-called *modified barrier function.* Let $\Omega := \{x \in \mathbb{R}^n : -\beta g(x) + 1 > 0\}$. The modified barrier function $L(x, \lambda) : \Omega \times \mathbb{R}_{>0} \to \mathbb{R}$, associated with (6), is defined by (see e.g. Pan (1990) or Polyak (1992))

$$L(x, \lambda) = h(x) - \lambda\beta^{-1} \log(-\beta g(x) + 1). \tag{8}$$

$L(x, \lambda)$ can thus be seen as a shifted barrier function since, for $\lambda \in \mathbb{R}_{>0}$, it tends to infinity as $g(x)$ tends to β^{-1}. Furthermore, $L(x, \lambda)$ can also be seen as the Lagrangian of the following constrained optimization problem

$$\min_{x\in\mathbb{R}^n} h(x) \text{ s.t. } \beta^{-1} \log(-\beta g(x) + 1) \geq 0, \tag{9}$$

which is equivalent to (6). Therefore, to solve problem (6), it is sufficient to steer x to a saddle point of (8) (see e.g. Rockafellar (1970)).

A way to achieve this objective is to implement the saddle point dynamics proposed in Dürr et al. (2013b),

$$\begin{cases} \dot{x} = -\rho_x \nabla_x L(x, \lambda) \\ \dot{\lambda} = \rho_\lambda \lambda \nabla_\lambda L(x, \lambda) = -\rho_\lambda \lambda \beta^{-1} \log(-\beta g(x) + 1) \end{cases}, \tag{10}$$

with $\rho_x \in \mathbb{R}_{>0}$ and $\rho_\lambda \in \mathbb{R}_{>0}$.

As stated in the following Theorem, system (10) also enforces (7):

Theorem 11. Let $\Omega := \{x \in \mathbb{R}^n : -\beta g(x) + 1 > 0\}$. Furthermore, let $\mathcal{S}_L \subset \Omega \times \mathbb{R}_{\geq 0}$ be the set of all saddle points of $L(x, \lambda)$, defined in (8). Assume that S_L is non-empty and compact. Then, under Assumptions 9, $\mathcal{S}_L$ is regionally uniformly asymptotically stable in $\Omega \times \mathbb{R}_{>0}$ for system (10).

Proof. The proof is given in Appendix C. ■

Remark 12. It can be seen from the proof of Theorem 11 that extending the approach to handle several constraints is not straightforwards. In particular, proving the positive invariance of x in Ω is not trivial, due to the possible interaction between the constraints.

Note that $\log(-\beta g(x)+1)$ in (10) is not defined for $g(x) \geq \beta^{-1}$. This may represent a drawback when $\nabla_x L(x, \lambda)$ is mathematically unknown, and has to be estimated, as it is the case in extremum seeking. Indeed, the estimated value of $\nabla_x L(x, \lambda)$ has to ensure that $g(x(t)) < \beta^{-1}, \forall t \geq t_0$. This may lead to a very restrictive set of admissible values for the extremum seeking parameters (e.g. ω). Furthermore, this set of admissible values is a priori unknown by the user. To alleviate this issue, we propose to introduce the factor $\left(\frac{-\beta g(x)+1}{1+(-\beta g(x)+1)}\right)^2$ in the dynamics of (10), namely we consider

$$\begin{cases} \dot{x} = -\rho_x \left(\dfrac{-\beta g(x) + 1}{1 + (-\beta g(x) + 1)}\right)^2 \nabla_x L(x, \lambda) \\ \dot{\lambda} = \rho_\lambda \lambda \left(\dfrac{-\beta g(x) + 1}{1+(-\beta g(x)+1)}\right)^2 \nabla_\lambda L(x, \lambda) \end{cases}, \tag{11}$$

with $\rho_x \in \mathbb{R}_{>0}$ and $\rho_\lambda \in \mathbb{R}_{>0}$. From (11), one may observe that, whatever the estimated value of $\nabla_x L(x, \lambda)$, if $g(x_0) < \beta^{-1}$, then $g(x(t)) < \beta^{-1}$, $\forall t \geq t_0$. In the next section, we will use this property to design an extremum seeking system, in the form of (2), whose associated Lie bracket system is (11), and that ensures the positive invariance of (x, λ) in $\Omega \times \mathbb{R}_{>0}$, for all $\omega \in \mathbb{R}_{>0}$.

The next Theorem precises the properties of system (11):

Theorem 13. Let $\Omega := \{x \in \mathbb{R}^n : -\beta g(x) + 1 > 0\}$. Furthermore, let $\mathcal{S}_L \subset \Omega \times \mathbb{R}_{\geq 0}$ be the set of all saddle points of $L(x, \lambda)$, defined in (8). Assume that S_L is non-empty and compact. Then, under Assumptions 9, the set $\mathcal{S}_L$ is regionally uniformly asymptotically stable in $\Omega\times\mathbb{R}_{>0}$ for system (11).

Proof. The only difference between systems (10) and (11) is the presence of the scaling factor $\left(\frac{-\beta g(x)+1}{1+(-\beta g(x)+1)}\right)^2$ in the dynamics of each state variable. It was proved in Theorem 11 that system (10) is positively invariant in $\Omega \times \mathbb{R}_{>0}$. Accordingly, at every point (x, λ) of the trajectory of system (10), the vector $[\dot{x}^T, \dot{\lambda}^T]^T$ of systems (10) and (11) have the same direction. Starting with the same initial conditions, the trajectories of the two systems ((10) and (11)) are therefore passing by the same values of (x, λ). The properties of stability, boundedness and convergence are thus identical for systems (10) and (11). ■

Since the saddle points of $L(x, \lambda)$ are the solutions of the optimization problem (6), Theorem 13 guarantees that the trajectories of system (11) asymptotically converge to the solution of (6), while fulfilling (7).

4.2 A Lie Bracket Approximation

In the previous section, we proposed the gradient-based system (11), that asymptotically converges to the solution of (6), while satisfying (7). However, the implementation of this system requires the knowledge of $\nabla_x L(x, \lambda)$. In this section, the Lie bracket formalism is exploited to design an extremum seeking system that approximates the trajectory of (11), by only combining time-periodic signals with the on-line measurements of $h(x)$ and $g(x)$.

The goal is therefore to select the vector fields $f_i(x)$ together with the signals $u_i(t)$ in (2) such that the associated Lie bracket system coincides with (11). As formally proved in the next Theorem, a possible extremum seeking system is

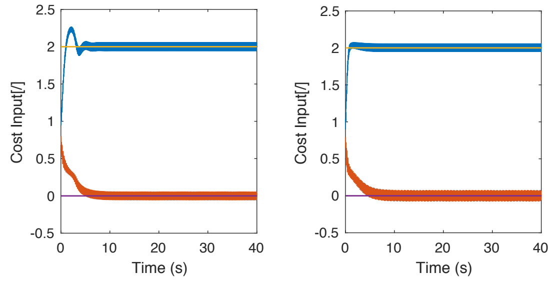
$$\begin{cases} \dot{x}_i = \sqrt{2\rho_x\omega k_i}\,\dfrac{-\beta g(x)+1}{1+(-\beta g(x)+1)}\cos(L(x,\lambda)+k_i\omega t) \\ \dot{\lambda} = -\rho_\lambda\lambda\left(\dfrac{-\beta g(x)+1}{1+(-\beta g(x)+1)}\right)^2 \dfrac{\log(-\beta g(x)+1)}{\beta} \end{cases}, \tag{12}$$

with $i \in \{1, 2, .., n\}$, $\omega \in \mathbb{R}_{>0}$, and $k_i \in \mathbb{Q}_{>0}$, with $k_i \neq k_j$ for $i \neq j$.

It can be noticed that (12) is positively invariant in $\Omega \times \mathbb{R}_{>0}$. In virtue of Lemma 7 and Theorem 13, this guarantees that the trajectory of the designed extremum seeking practically asymptotically converges to the solution of (6), while satisfying (7).

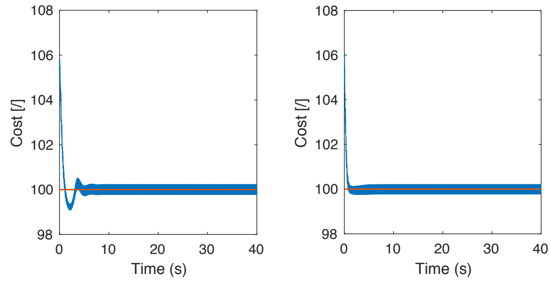
The stability property of system (12) is formalized in the following Theorem:

Theorem 14. Consider system (12). Let $\Omega := \{x \in \mathbb{R}^n : -\beta g(x) + 1 > 0\}$, and suppose that Assumption 9 holds. Then, the Lie-bracket system associated with (12) is (11). Furthermore, let $S_L \subset \Omega \times \mathbb{R}_{>0}$ be the set of all saddle points of $L(x, \lambda)$, defined in (8), and suppose that S_L is non-empty and compact. Then, under Assumption 9, S_L is semi-regionally practically uniformly asymptotically stable in $\Omega\times\mathbb{R}_{>0}$ for system (12), with the parameter ω^{-1}.



(a) $\beta = 0.5$ (b) $\beta = 2$

Fig. 1. Simulation of the extremum seeking system (12) with the constrained optimization problem (13): Evolution of $x_1(t)$ (—) and $x_2(t)$ (—), Solution of the optimization problem (13) x_1^* (—) and x_2^* (—).



(a) $\beta = 0.5$ (b) $\beta = 2$

Fig. 2. Simulation of the extremum seeking system (12) with the constrained optimization problem (13): Evolution of $h(x(t))$ (—), Solution of the optimization problem (13) h^* (—).

Proof. The proof is reported in Appendix D.

5. SIMULATION RESULTS

To illustrate the effectiveness of the proposed extremum seeking system (12), we consider the constrained optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^2} h(x) &:= (x_1 - 4)^2 + x_2^2 + 96 \\ \text{s.t. } g(x) &:= x_1^2 + x_2^2 - 4 \leq 0, \end{aligned} \tag{13}$$

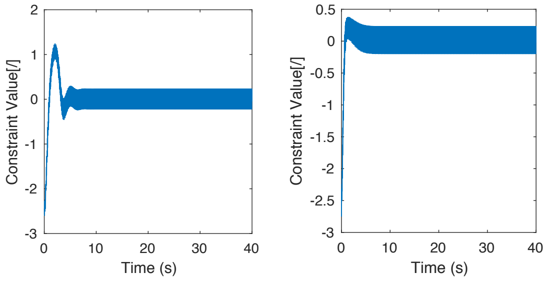
whose solution is $h^* = 100$, with $(x_1^*, x_2^*) = (2, 0)$.

To perform the simulation, the following parameters are selected: $\omega = 100 rad/s$, $k_1 = 1$, $k_2 = 1.3$, $\rho_x = 0.64$, $\rho_\lambda = 10$, $x_0 = [1,\ 0.75]^T$ and $\lambda_0 = 5$.

In Figure 1, it can be observed that the cost inputs converge in a neighborhood of the constrained minimizer $(x_1^*, x_2^*) = (2, 0)$. Accordingly, one can see in Figure 2 that the cost converges in a neighborhood of its constrained minimum, i.e. $h^* = 100$. In Figure 3, one can also verify that the constraint is never violated of more than β^{-1}. In agreement with Theorem 14, one can observe that, the larger the value of β, the smaller the violation of the constraint.

5.1 Non-Steady State Behavior

To conclude the study of the proposed approach, let us now consider that the cost function is associated with a dynam-



(a) $\beta = 0.5$ (b) $\beta = 2$

Fig. 3. Simulation of the extremum seeking system (12) with the constrained optimization problem (13): Evolution of the constraint $g(x(t))$ (—).

ical system. Namely, consider the following constrained optimization problem

$$\begin{aligned} &\min_{\chi \in \mathbb{R}^2} c(\chi) := (0.6\chi_1^3 - 4)^2 + 0.25\chi_2^6 + 96 \\ &\text{s.t. } \gamma(\chi) := 0.36\chi_1^6 + 0.25\chi_2^6 - 4 < 0, \end{aligned} \tag{14}$$

associated with the dynamical system

$$\begin{cases} \dot{\chi}_1 &= -150\chi_1^3 + 250x_1 \\ \dot{\chi}_2 &= -50\chi_2^3 + 100x_2 \end{cases}. \tag{15}$$

The aim is now to steer system (15) to a steady-state value that solves (14), while ensuring that $\gamma(\chi(t)) \leq \beta^{-1}$, $\forall t \geq t_0$. Note that the steady-state optimization problem corresponding to (14)-(15) is nothing but (13).

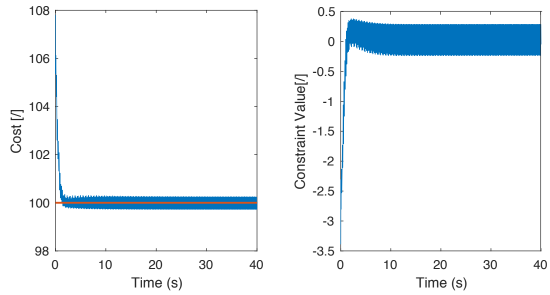
To achieve this objective, we will use (12) with $\gamma(\chi)$ and $c(\chi)$ instead of $g(x)$ and $h(x)$, respectively. Furthermore, we will introduce a small parameter μ in (12) to ensure a time-scale separation between the χ-dynamics and the extremum seeking system. Namely, we will consider the extremum-seeking system

$$\begin{cases} \dot{x}_i = \mu\sqrt{2\rho_x\omega k_i}\dfrac{-\beta\gamma(\chi)+1}{1+(-\beta\gamma(\chi)+1)}\cos(L_1(\chi,\lambda)+\mu k_i\omega t) \\ \dot{\lambda} \ = -\mu\rho_\lambda\lambda\left(\dfrac{-\beta\gamma(\chi)+1}{1+(-\beta\gamma(\chi)+1)}\right)^2\dfrac{\log(-\beta\gamma(\chi)+1)}{\beta}, \end{cases} \tag{16}$$

with $i \in \{1,2\}$ and $L_1(\chi,\lambda) := c(\chi) - \lambda\beta^{-1}\log(-\beta\gamma(\chi)+1)$.

Although the rigorous analysis of this case is outside the scope of this paper, the following reasoning is possible. For μ sufficiently small, the x and λ-dynamics are much slower than the χ-dynamics. Therefore, if χ_0 is close to its steady state value $l(x_0)$, $\chi(t)$ will remain close to its steady state value $l(x(t))$, at all time. Hence, $\gamma(\chi(t)) \approx g(x(t))$ and $L_1(\chi(t),\lambda(t)) \approx L(x(t),\lambda(t))$, at all time. The trajectory of system (16) will thus approximate the trajectory of system (12), with the change of time scale $t' = \mu t$. Since the trajectory of system (12) is positively invariant in the feasible region Ω, this implies that, by selecting μ sufficiently small, the x-trajectory of system (16) will also remain in Ω and practically asymptotically converge to the solution of (13). Furthermore, if χ_0 is sufficiently close to its steady state value and μ is sufficiently small, $||\gamma(\chi(t)) - g(x(t))||$ will be sufficiently small to result in $\gamma(\chi(t)) < \beta^{-1}$, $\forall t \geq t_0$.

To perform the simulation, we use the same parameters as the ones used for the static case, and we select $\chi_0 = [1,1]^T$ and $\mu = 0.5$. In Figure 4, it can be noticed that the



(a) Evolution of $c(\chi(t))$ (—) and solution of (13) (—) (b) Evolution of $\gamma(\chi(t))$ for $\beta = 2$

Fig. 4. Simulation of the extremum seeking system (16) with the constrained optimization problem (14)-(15).

constraint is never violated more than β^{-1} and the cost converges in a neighborhood of its steady-state constrained minimum, i.e. $h^* = 100$. As expected, the trajectories of $c(\chi(t))$ and $\gamma(\chi(t))$ are close to the ones of $h(x(t))$ and $g(x(t))$, respectively (cf. Figures 2 and 3).

6. CONCLUSION

In this paper, we designed a novel extremum seeking system to solve constrained optimization problems. As a first result, we obtain systems (10) and (11), based on a modified-barrier function and a saddle point dynamics. We showed in Theorems 11 and 13 that the proposed systems satisfy the maximal violation of the constraint (7) for all time. As a second result, the Lie bracket approximation theory was used to approximate (11), by combining time-periodic signals with the on-line measurement of both the cost and the constraint. The obtained extremum seeking scheme (12) was proved in Theorem 14 to be able to steer the cost input towards the value that minimizes the cost under a single convex constraint, while allowing to define the maximal violation of the constraints during the whole transient phase. Its effectiveness has also been demonstrated in simulation. Future works will notably consider the case of multiple constraints and aim to improve the steady-state performances, by making use of adaptive dither amplitudes.

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Appendix A. PROOF OF LEMMA 6

The difference between the assumptions of Lemma 6 in this paper and Lemma 1.1 in Labar et al. (2018a) is that the vector fields are only defined, and of class C^2, in a region $\mathcal{R} \subseteq \mathbb{R}^n$, instead of being defined, and of class C^2, in $\mathbb{R}^n$. However, since we assumed that the input-affine system is positively invariant in $\mathcal{R}$, the Chen-Fliess expansion of its trajectory can be performed as in (22)-(38), Appendix A of Labar et al. (2018a). Furthermore, since the trajectory of the Lie bracket system (3) is assumed to be absolutely continuous, there exists a bounded set $\mathcal{W}_1 \subseteq \mathcal{R}$ that contains the trajectory of system (3) on the interval $t \in [0, t_f]$, for all $\overline{x}_0 \in \mathcal{V}$. The set $\mathcal{R}$ being convex, there exists thus a bounded convex set $\mathcal{W} : \mathcal{W}_1 \subseteq \mathcal{W} \subseteq \mathcal{R}$ such that $\overline{x}(t) \in \mathcal{W}, \forall t \in [0, t_f]$. Let $\mathcal{W}_x : U_{\mathcal{R}}^{\mathcal{W}}(D)$. Since $\forall i, j \in \{0, 1, ..., l\}$, the vector fields $f_i(x)$ and the Jacobian matrices $\frac{\partial f_i(x)}{\partial x}$ and $\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right)$ are bounded on every open bounded set $\mathcal{A} \subseteq \mathcal{R}$, the different remainders appearing in the Chen-Fliess expansion can be bounded as in (41)-(52), Appendix A of Labar et al. (2018a) for all $x \in \mathcal{W}_x$. Finally, since $\mathcal{W}_x$ is a convex set, the assumption that $\forall i, j \in \{0, 1, ..., l\}$, the Jacobian matrices $\frac{\partial f_i(x)}{\partial x}$ and $\frac{\partial}{\partial x}\left(\frac{\partial f_i(x)}{\partial x} f_j(x)\right)$ are bounded on every open bounded set $\mathcal{A} \subseteq \mathcal{R}$ implies that the vector fields $f_i(x)$ and the Lie brackets $[f_i(x), f_j(x)]$ are locally Lipschitz on $\mathcal{W}_x$. Thus, the bounds obtained in (53)-(56), Appendix A of Labar et al. (2018a) also hold, and the closeness of trajectories (5) is therefore obtained. ■

Appendix B. PROOF OF LEMMA 7

To prove Lemma 7, we refer to the three conditions of sRPUAS given in Definition 4. The proof follows thus a similar reasoning to the proof of Lemma 2 in Dürr et al. (2013b). The difference comes from the use of a slightly more general definition of sRPUAS.

Let $\delta_V \in \mathbb{R}_{>0}$ and $\delta_B \in \mathbb{R}_{>0}$ be arbitrary, but fixed.

Stability. Let $\delta_B' \in (0, \delta_B)$. From the stability property of the Lie bracket system (3), there exists a $\delta_Q' \in \mathbb{R}_{>0}$ such that, for every $t_0 \in \mathbb{R}$, if $\overline{x}(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q')$ then $\overline{x}(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B')$, $\forall t \geq t_0$.
Select now $\delta_Q'' \in (0, \delta_Q')$. From the uniform asymptotic convergence of the Lie bracket system (3), there exists a time $t_1 \in \mathbb{R}_{>0}$ such that, for all $t_0 \in \mathbb{R}$, if $\overline{x}(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q')$ then $\overline{x}(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q'')$, $\forall t \geq t_0 + t_1$. Let $\omega_S^* \in \mathbb{R}_{>0}$ come from Lemma 6, with $t_f = t_1$, $\mathcal{V} = U_{\mathcal{R}}^{\mathcal{S}}(\delta_{Q'})$ and $D = \min\left\{\delta_B - \delta_B', \delta_Q' - \delta_Q''\right\}$. The positive invariance of system (2) in $\mathcal{R}$ allows then to conclude that, for all $\omega \in (\omega_S^*, \infty)$ and $t_0 \in \mathbb{R}$, if $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q')$, then i)$x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, for all $t \in [t_0, t_0 + t_f]$ and ii)$x(t_0 + t_f) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q')$. Since $x(t_0 + t_f) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q')$, the same reasoning can be conducted again, implying that with the obtained values of ω, it holds $x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B), \forall t \geq t_0$. Therefore the stability part is concluded with $\delta_Q := \delta_Q'$.

Boundedness. From the boundedness property of the Lie bracket system (3), there exists a $\delta_{W'} \in \mathbb{R}_{>0}$ such that, for every $t_0 \in \mathbb{R}$, if $\overline{x}(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$ then $\overline{x}(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_W')$, $\forall t \geq t_0$.
Let $\delta_{V'} \in (0, \delta_V)$. From the uniform asymptotic convergence of the Lie bracket system (3), there exists a time $t_1 \in \mathbb{R}_{>0}$ such that, for all $t_0 \in \mathbb{R}$, if $\overline{x}(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$ then $\overline{x}(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_{V'})$, $\forall t \geq t_0 + t_1$.
Let ω_B^* come from Lemma 6, with $t_f = t_1$, $\mathcal{V} = U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$ and $D = \delta_V - \delta_{V'}$. The positive invariance of system (2) in $\mathcal{R}$ allows then to conclude that, for all $\omega \in (\omega_B^*, \infty)$ and $t_0 \in \mathbb{R}$, if $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$, then i)$x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_W' + D)$, for all $t \in [t_0, t_0 + t_f]$ and ii)$x(t_0 + t_f) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$. Since $x(t_0 + t_f) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$, the same reasoning can be conducted again, implying that with the obtained values of ω, it holds $x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_W' + D), \forall t \geq t_0$. Therefore the boundedness part is concluded by selecting $\delta_W := \delta_W' + D$.

Practical Convergence. From the **Stability** part, we know that there exist a $\delta_Q \in \mathbb{R}_{>0}$ and an $\omega_S^* \in \mathbb{R}_{>0}$ such that, for every $t_0 \in \mathbb{R}$ and $\omega \in (\omega_S^*, \infty)$, if $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q)$ then $x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, $\forall t \geq t_0$.
Let $\delta_{Q'} \in (0, \delta_Q)$. From the uniform asymptotic convergence of the Lie bracket system (3), there exists a time $t_1 \in \mathbb{R}$ such that, for every $t_0 \in \mathbb{R}$, if $\overline{x}(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$ then $\overline{x}(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_{Q'})$, $\forall t \geq t_0 + t_1$. Let ω_C^* come from Lemma 6, with $t_f = t_1$, $\mathcal{V} = U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$ and $D = (\delta_Q - \delta_{Q'})$. The positive invariance of system (2) in $\mathcal{R}$ allows then to conclude that, for all $\omega \in (\max\{\omega_C^*, \omega_S^*\}, \infty)$, if $x(t_0) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_V)$, then $x(t_0 + t_1) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_Q)$ and hence $x(t) \in U_{\mathcal{R}}^{\mathcal{S}}(\delta_B)$, for all $t \geq t_0 + t_1$.

The proof is therefore concluded by selecting $\omega^* := \max\{\omega_S, \omega_B, \omega_C\}$. ■

Appendix C. PROOF OF THEOREM 11

To perform the proof of Theorem 11, we proceed in two steps. In *Step 1*, system (10) is proved to be positively invariant in $\Omega \times \mathbb{R}_{>0}$. In *Step 2*, a Lyapunov function candidate $V(x,\lambda) : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is introduced. It is shown that $\dot{V}(x,\lambda) \leq 0, \forall (x,\lambda) \in \Omega \times \mathbb{R}_{>0}$ and the LaSalle's invariance principle is used to conclude the regional asymptotic convergence of (x,λ) to S_L.

Step 1. The positive invariance of λ in $\mathbb{R}_{>0}$ can be directly deduced from the dynamics of system (10). To prove the positive invariance of x in Ω, let us consider the function $V_I : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined as follows:

$$V_I(x) := 0.5(-\beta g(x) + 1)^2. \tag{C.1}$$

Its time-derivative, along the trajectory of system (10), is

$$\dot{V}_I(x,\lambda) = -\beta(-\beta g(x) + 1)\nabla^T g(x) \times \left[-\rho_x \nabla h(x) - \rho_x \lambda \frac{\nabla g(x)}{-\beta g(x) + 1}\right], \tag{C.2}$$

or equivalently,

$$\dot{V}_I(x,\lambda) = \beta\rho_x \left[\lambda \left|\left|\nabla g(x)\right|\right|^2 + (-\beta g(x) + 1)\nabla^T g(x)\nabla h(x)\right]. \tag{C.3}$$

Let $\mathcal{E} := \{x \in \mathbb{R}^n : 0 < -\beta g(x) + 1 < 0.5\}$. The x-trajectory is continuous for system (10). Therefore, starting initially in Ω, x cannot reach $-\beta g(x) + 1 = 0$, without entering in $\mathcal{E}$. Therefore, to prove the positive invariance of x in Ω for system (10), it is sufficient to show that x cannot reach $-\beta g(x) + 1 = 0$, starting in $\mathcal{E}$.

From Assumption 9, the functions $h(x)$ and $g(x)$ belong to class C^2. Accordingly, since $\mathcal{E}$ is a bounded set, there exists a $M_G \in \mathbb{R}_{>0}$ such that $\sup_{x\in\mathcal{E}} \left|\left|\nabla^T g(x)\nabla h(x)\right|\right| < M_G$. Furthermore, since $g(x)$ is convex (Assumption 9), and since the optimization problem (6) is feasible, $\nabla g(x)$ cannot vanish [1] in $\overline{\mathcal{E}}$. Accordingly, there exists a $m_g \in \mathbb{R}_{>0}$ such that $\inf_{x\in\mathcal{E}} \left|\left|\nabla g(x)\right|\right|^2 > m_G$. Therefore, (C.3) can be lower bounded as

$$\dot{V}_I(x,\lambda) \geq \beta\rho_x \left[\lambda m_G - (-\beta g(x) + 1)M_G\right], \tag{C.4}$$

$\forall x \in \mathcal{E}$, $\lambda \in \mathbb{R}_{>0}$. Accordingly, it holds that, if $(x,\lambda) \in \mathcal{E} \times \mathbb{R}_{>0}$ and $-\beta g(x) + 1 < \lambda m_G M_G^{-1}$, then $\dot{V}_I(x,\lambda) > 0$. Since λ is positively invariant in $\mathbb{R}_{>0}$ and $\dot{\lambda}(t) > 0$ for $x \in \mathcal{E}$, one can thus conclude that x cannot reach $-\beta g(x) + 1 = 0$, starting in $\mathcal{E}$, concluding the positive invariance of system (10) in $\Omega \times \mathbb{R}_{>0}$.

Step 2. Let $(x^*, \lambda^*) \in S_L$. Inspired by Dürr et al. (2013b) and Grushkovskaya and Ebenbauer (2016), the following Lyapunov function candidate $V(x,\lambda) : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ is considered

$$V(x,\lambda) = 0.5\rho_x^{-1}(x - x^*)^T(x - x^*) + \rho_\lambda^{-1} T(\lambda, \lambda^*), \tag{C.5}$$

with

$$T(\lambda,\lambda^*) = \begin{cases} \lambda & \text{if } \lambda^* = 0 \\ \lambda - \lambda^* + \lambda^* \log\left(\dfrac{\lambda^*}{\lambda}\right) & \text{if } \lambda^* > 0 \end{cases}. \tag{C.6}$$

Its time-derivative, along the trajectory of system (10), is

$$\dot{V}(x,\lambda) = -(x - x^*)^T \nabla_x L(x,\lambda) + (\lambda - \lambda^*)\nabla_\lambda L(x,\lambda). \tag{C.7}$$

From Assumption 9, the functions $h(x)$ and $g(x)$ are strictly convex and convex, respectively. Therefore, the following inequality holds (cf. first order Taylor expansion)

$$L(x^*,\lambda) \geq L(x,\lambda) + (x^* - x)^T \nabla_x L(x,\lambda), \tag{C.8}$$

$\forall x \in \Omega, \lambda \in \mathbb{R}_{\geq 0}$. Furthermore, since $L(x,\lambda)$ is affine in λ, one has

$$L(x,\lambda^*) = L(x,\lambda) + (\lambda^* - \lambda)\nabla_\lambda L(x,\lambda), \tag{C.9}$$

$\forall x \in \Omega, \lambda \in \mathbb{R}_{\geq 0}$. Substituting (C.8) and (C.9) in (C.7) yields

$$\dot{V}(x,\lambda) \leq L(x^*,\lambda) - L(x,\lambda^*), \tag{C.10}$$

$\forall x \in \Omega, \lambda \in \mathbb{R}_{>0}$. Since (x^*,λ^*) is a saddle point of $L(x,\lambda)$, it also holds (cf. Definition 1)

$$L(x^*,\lambda) \leq L(x^*,\lambda^*) \leq L(x,\lambda^*), \forall x \in \Omega, \lambda \in \mathbb{R}_{\geq 0}. \tag{C.11}$$

Accordingly, $\dot{V}(x,\lambda) \leq 0, \forall x \in \Omega, \lambda \in \mathbb{R}_{>0}$. Combining this result with the compactness of the level sets of $V(x,\lambda)$ and the positive invariance of system (10) in $\Omega \times \mathbb{R}_{>0}$ concludes the stability and boundedness of S_L in $\Omega \times \mathbb{R}_{>0}$ for system (10) (see Definition 4). To prove the regional asymptotic convergence of system (10) to S_L, we will refer to the LaSalle's invariance principle and show that the set of points $(x,\lambda) \in \Omega \times \mathbb{R}_{>0}$ such that $\dot{V}(x,\lambda) = 0$ do not contain any complete trajectory of system (10), except S_L.

Since $h(x)$ and $g(x)$ are strictly convex and convex, respectively, the function $L(x,\lambda^*)$ is strictly convex. It results thus $L(x^*,\lambda^*) < L(x,\lambda^*)$, $\forall x \in \Omega\backslash\{x^*\}$. Accordingly, referring to (C.11), $L(x^*,\lambda) = L(x,\lambda^*)$ implies that $x = x^*$. Therefore, the set of points $(x,\lambda) \in \Omega \times \mathbb{R}_{>0}$ such that $\dot{V}(x,\lambda) = 0$ satisfy the equality $L(x^*,\lambda) = L(x^*,\lambda^*)$, namely $\lambda \log(-\beta g(x^*) + 1) = \lambda^* \log(-\beta g(x^*) + 1)$. Since $(x^*,\lambda^*) \in \mathcal{S}_L$, the previous equality reduces to $\lambda \log(-\beta g(x^*) + 1) = 0$. Therefore, the set of points $(x,\lambda) \in \Omega \times \mathbb{R}_{>0}$ such that $\dot{V}(x,\lambda) = 0$ is $S_V := \{(x,\lambda) \in \Omega \times \mathbb{R}_{>0} : i) x = x^*; ii) \lambda \log(-\beta g(x^*) + 1) = 0\}$. Referring to system (10), the only complete trajectories in S_V are such that $\nabla_x L(x^*,\lambda) = 0$, and hence are the points satisfying the KKT conditions (Definition 2), which are precisely the saddle points of $L(x,\lambda)$ (Rockafellar, 1970), concluding the proof. ■

Appendix D. PROOF OF THEOREM 14

In this section, the proof of Theorem 14 is carried out. To do so, three main steps are followed. In *Step 1*, system

[1] Otherwise, it would hold $g(x) \geq 0.5\beta^{-1}, \forall x \in \Omega$, which contradicts the feasibility of the optimization problem

(12) is rewritten in the form of (2). Referring to (2)-(3), it is then shown, in *Step 2*, that the Lie bracket system associated with (12) is (11). In *Step 3*, the stability property of system (12) is finally proved, by combining Lemma 7 with Theorem 13.

Step 1. Using the trigonometric identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, one may rewrite the x-dynamics of (12) as

$$\dot{x} = \sum_{i=1}^{2n} \sqrt{\omega} F_i(x) u_i(k_i \omega t), \tag{D.1}$$

with

$$u_i(t) = \begin{cases} \cos(t) & 1 \leq i \leq n \\ \sin(t) & n < i \leq 2n \end{cases}, \tag{D.2}$$

$$k_i = k_{i-n} \quad \text{for} \quad n < i \leq 2n, \tag{D.3}$$

and

$$F_i(x) = \begin{cases} \sqrt{2\rho_x k_i} \dfrac{-\beta g(x)+1}{1+(-\beta g(x)+1)} \cos(L(x,\lambda)) e_i, \\ \qquad\qquad 1 \leq i \leq n \\ -\sqrt{2\rho_x k_i} \dfrac{-\beta g(x)+1}{1+(-\beta g(x)+1)} \sin(L(x,\lambda)) e_{i-n}, \\ \qquad\qquad n < i \leq 2n \end{cases}. \tag{D.4}$$

Taking this result into account, (12) can be written in the form of (2), namely

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = f_0(x,\lambda) + \sum_{i=1}^{2n} \sqrt{\omega} f_i(x,\lambda) u_i(k_i \omega t), \tag{D.5}$$

with

$$f_0(x,\lambda) = -\rho_\lambda \lambda \left(\frac{-\beta g(x)+1}{1+(-\beta g(x)+1)} \right)^2 \frac{\log(-\beta g(x)+1)}{\beta} e_{n+1}, \tag{D.6}$$

and $f_i(x,\lambda) = [F_i^T(x), 0]^T$.

Step 2. From (2)-(3), the Lie bracket system associated with (D.5) (and hence with (12)) is given by

$$\begin{bmatrix} \dot{\overline{x}} \\ \dot{\overline{\lambda}} \end{bmatrix} = f_0(\overline{x},\overline{\lambda}) + \sum_{\substack{1 \leq i < 2n \\ i < j \leq 2n}} [f_i(\overline{x},\overline{\lambda}), f_j(\overline{x},\overline{\lambda})] \gamma_{ij}. \tag{D.7}$$

Referring to the definition of γ_{ij}, given in (4), together with (D.2) and (D.3), one has

$$\gamma_{ij} = \begin{cases} 0.5 k_i^{-1} & 1 \leq i \leq n, j = i+n \\ 0 & \text{else} \end{cases}. \tag{D.8}$$

Accordingly, the Lie bracket system associated with (12) is

$$\begin{bmatrix} \dot{\overline{x}} \\ \dot{\overline{\lambda}} \end{bmatrix} = f_0(\overline{x},\overline{\lambda}) + 0.5 k_i^{-1} \sum_{1 \leq i \leq n} [f_i(\overline{x},\overline{\lambda}), f_{i+n}(\overline{x},\overline{\lambda})]. \tag{D.9}$$

From (D.4), one gets

$$[f_i, f_{i+n}](\overline{x},\overline{\lambda}) = -2\rho_x k_i \left(\frac{-\beta g(\overline{x})+1}{1+(-\beta g(\overline{x})+1)} \right)^2 \frac{\partial L(\overline{x},\overline{\lambda})}{\partial \overline{x}_i} e_i. \tag{D.10}$$

Therefore (D.9) reduces to

$$\begin{cases} \dot{\overline{x}} = -\rho_x \left(\dfrac{-\beta g(\overline{x})+1}{1+(-\beta g(\overline{x})+1)} \right)^2 \nabla_x L(\overline{x},\overline{\lambda}) \\ \dot{\overline{\lambda}} = -\rho_\lambda \overline{\lambda} \left(\dfrac{-\beta g(\overline{x})+1}{1+(-\beta g(\overline{x})+1)} \right)^2 \dfrac{\log(-\beta g(\overline{x})+1)}{\beta} \end{cases}, \tag{D.11}$$

which is nothing but (11), concluding the first part of the statement.

Step 3. We know from Theorem 13 that S_L is regionally uniformly asymptotically stable in $\Omega \times \mathbb{R}_{>0}$ for system (11). Since $\Omega \times \mathbb{R}_{>0}$ is positively invariant for system (12), Lemma 7 concludes the semi-regional uniform practical asymptotic stability of S_L in $\Omega \times \mathbb{R}_{>0}$ for system (12), with the parameter ω^{-1}. ∎