Improving Performance in Robust Economic MPC Using Stochastic Information [⋆]

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Abstract: In this paper, we develop a new tube-based robust economic MPC scheme for linear systems subject to bounded disturbances with given distributions. By using the error distribution in the predictions of the finite horizon optimal control problem, we can incorporate stochastic information in order to improve the expected performance while being able to guarantee strict feasibility. For this new framework, we can provide bounds on the asymptotic average performance of the closed-loop system. Moreover, a constructive approach is sketched in order to find an appropriate quadratic terminal cost accepting a slight degradation of the average performance statement.

Keywords: Robust MPC, Economic MPC

1. INTRODUCTION

In the last decade, *economic* Model Predictive Control (MPC) has received significant attention. While stabilizing MPC uses a positive definite stage cost function to stabilize a given set-point, the main focus of economic MPC is the optimization of some general performance criterion, which possibly resembles the economics of the considered system. To this end, different settings and methods have been proposed in the literature (see, e.g., Angeli et al. (2012), Diehl et al. (2011), Amrit et al. (2011), Müller et al. (2013)).

In many practical applications, systems are affected by disturbances which can result in a degradation of performance and/or a loss of feasibility. Therefore, different concepts have been presented in the framework of MPC in order to deal with disturbances. In robust MPC, bounded disturbances are taken into consideration while aiming at the robust satisfaction of hard constraints (see, e.g., Chisci et al. (2001), Mayne et al. (2005)). In stochastic MPC, disturbances of stochastic nature, i.e., with a given distribution, are considered. This stochastic information can be used in order to improve performance. Moreover, probabilistic constraints are typically considered instead of hard constraints (see, e.g., de la Peña et al. (2005), Cannon et al. (2009), Primbs and Sung (2009), Chatterjee et al. (2011)).

When considering economic stage cost functions for disturbed systems, only few results can be found. In Huang et al. (2012), a stability result for robust economic MPC is presented which is based on a robust tracking of an a priori determined optimal nominal trajectory. Broomhead et al. (2014) aim at stabilizing an economically optimal steady-state despite disturbances. Lucia et al. (2014) study a scenario based approach for economic MPC in order to, among others, find guarantees for feasibility. Another approach is presented in Bayer et al. (2014), where a robust MPC framework is employed to guarantee robust constraint satisfaction. In order to consider the influence of the disturbances on the performance, the cost function is modified by averaging the cost over all possible states within some invariant set. Since no further assumption on the disturbances is imposed other than boundedness, this averaging is done by weighting all states in the invariant set equally.

In this paper, we show how additional stochastic information on the disturbance, if available, can be used to improve closed-loop performance in robust economic MPC. To this end, we consider a robust MPC framework to guarantee robust constraint satisfaction, and employ additional stochastic information within the cost function to improve the performance. To this end, we compute the exact prediction of the error set at each open-loop time step and use the robust MPC approach presented in Chisci et al. (2001) to guarantee robust feasibility. Moreover, we compute the distribution of the error over these sets using the given distribution of the disturbance, and employ this information within the finite horizon optimal control problem by taking the expected value of the cost. We show that for a particular assumption on the terminal cost, bounds on the average performance of the closed loop can be derived which resemble known results from both nominal economic MPC and previous concepts on robust economic MPC. Moreover, we derive a constructive approach for finding an appropriate quadratic terminal cost (which leads to a slight degradation of the original average performance statement).

The remainder of this paper is structured as follows. In Section 2, we introduce the problem setup. The finite horizon optimal control problem is presented and discussed in Section 3. A bound on the closed-loop asymptotic average performance is derived in Section 4, and in Section 5, we provide a constructive approach for finding an appropriate quadratic terminal cost. The setup is applied to a numerical example from process industry in Section 6, and the paper is concluded in Section 7.

Notation: We denote by $\mathbb{I}_{\geq 0}$ the set of all non-negative integers and by $\mathbb{I}_{[a,b]}$ the set of all integers in the interval $[a, b] \subseteq \mathbb{R}$.

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For sets $X, Y \subseteq \mathbb{R}^n$, the Minkowski set addition is defined by $X \oplus Y := \{x + y \in \mathbb{R}^n : x \in X, y \in Y\}$; the Pontryagin set difference is defined as $X \ominus Y := \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$.

2. PROBLEM SETUP

In this paper, we consider discrete-time LTI systems of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t), \tag{1}$$

where $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$ is the system state and $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ is the input at time $t \in \mathbb{I}_{\geq 0}$, respectively. For the states and inputs, we consider pointwise-in-time constraints of the form $(x(t), u(t)) \in \mathbb{Z}$, for all $t \in \mathbb{I}_{\geq 0}$, where $\mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$ is a compact set. We assume that (A, B) is stabilizable.

The unknown disturbance $w(t)$ at time t satisfies the following assumption.

Assumption 1. For each $t \in \mathbb{I}_{\geq 0}$, the disturbance satisfies

$$w(t) \in \mathbb{W} \subset \mathbb{R}^n, \tag{2}$$

where $\mathbb{W}$ is a C-set, that is, it is a compact and convex set containing the origin in its interior. Furthermore, w is distributed over $\mathbb{W}$ according to some given probability density function (PDF) $\rho_{\mathbb{W}} : \mathbb{R}^n \to [0, \infty]$, which has bounded support $\mathbb{W}$. All disturbances are *identical and independently distributed* (i.i.d.) and have *zero mean*. ■

Note that due to the bounded support, $\rho_{\mathbb{W}}(w) \neq 0$ only if $w \in \mathbb{W}$. Furthermore, we know that

$$\int_{\mathbb{R}^n} \rho_{\mathbb{W}}(w)\mathrm{d}w = \int_{\mathbb{W}} \rho_{\mathbb{W}}(w)\mathrm{d}w = 1. \tag{3}$$

For the input u, we employ an affine parametrization of the form

$$u(t) = Kx(t) + c(t), \tag{4}$$

where $c(t) \in \mathbb{R}^m$ is the manipulated input at time $t \in \mathbb{I}_{\geq 0}$ and $K \in \mathbb{R}^{m \times n}$ is a state feedback, determined such that $A_{\text{cl}} = A + BK$ is a stable matrix.

Thus, system (1) can equivalently be written as

$$x(t+1) = A_{\text{cl}}x(t) + Bc(t) + w(t). \tag{5}$$

For A_{cl}, we introduce the following assumption.

Assumption 2. The system matrix A_{cl} is invertible. ■

Remark 3. In principle, this assumption could be relaxed. However, for ease of presentation, we restrict ourselves to system matrices which are invertible. Note that this is not a major restriction. In fact, if (A, B) is controllable, the eigenvalues of A_{cl} can be placed arbitrarily inside the unit disc. Note that the state feedback K is later used to limit the prediction errors in order to prevent them from growing exponentially. ■

Our objective in the following is to find a feasible control input to system (1) minimizing the asymptotic average performance

$$\limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ell(x(t), u(t)), \tag{6}$$

where $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ can be some general stage cost function which is assumed to be continuous.

2.1 Invariant Error Set and Error Dynamics

As the real system state can not be predicted exactly due to the disturbance, we adapt the idea of a nominal system from the framework of tube-based MPC. Within the predictions, we make use of the nominal state by neglecting the disturbances. Hence the predicted nominal system is given by

$$z(k+1|t) = A_{\text{cl}}z(k|t) + Bc(k|t), \;\; z(0|t) = x(t). \tag{7}$$

The notation $z(k|t)$ and $c(k|t)$ denotes k-step ahead predictions of states and inputs, predicted at time $t \in \mathbb{I}_{\geq 0}$. By means of the nominal system, we can introduce the predictions for the error

$$e(k|t) = x(k+t) - z(k|t) \tag{8}$$

as the difference between the real and the predicted nominal system with its associated dynamics

$$e(k+1|t) = A_{\text{cl}}e(k|t) + w(k+t), \; e(0|t) = 0. \tag{9}$$

The initial condition follows as the prediction for the nominal system at time t is initialized at the real state, i.e., $z(0|t) = x(t)$. The error at prediction time $k \in \mathbb{I}_{\geq 0}$ is contained in the set Ω_k, defined recursively as

$$\Omega_{k+1} = A_{\text{cl}}\Omega_k \oplus \mathbb{W}, \tag{10}$$

with $\Omega_0 = \{0\}$. It is a known result in literature (see, e.g. Raković et al. (2005)) that for a stable system, this recursion converges to the minimal robust positively invariant set Ω_∞.

Definition 4. (Blanchini (1999)). The set $\Omega_\infty \subset \mathbb{R}^n$ is a robust positively invariant (RPI) set for system (9) if for all $e(0|t) \in \Omega_\infty$ and all $w(k+t) \in \mathbb{W}$ the solution is such that $e(k|t) \in \Omega_\infty$ for all $k \in \mathbb{I}_{>0}$. ■

The notation Ω_∞ stems from the fact that the minimal RPI (mRPI) set can – in theory – be understood as $\Omega_\infty = \bigoplus_{i=0}^{\infty} A_{\text{cl}}^i \mathbb{W}$. Moreover, by definition, the mRPI set satisfies $A_{\text{cl}}\Omega_\infty \oplus \mathbb{W} \subseteq \Omega_\infty$.

2.2 Error Distributions

Next, we derive the distribution of the error over the error sets Ω_k. From (9), the error dynamics are known and we can use concepts from probability theory to find the probability density function (PDF) of the error. It is well known (see, e.g., Ross (2006)) that the summation of two random variables $X, Y \in \mathbb{R}^n$ with their associated PDFs $f_X(\epsilon)$ and $f_Y(\epsilon)$ results in a random variable with the PDF

$$f_{X+Y}(\epsilon) = \int_{\mathbb{R}^n} f_X(\epsilon - y) f_Y(y)\mathrm{d}y =: (f_X * f_Y)(\epsilon). \tag{11}$$

Moreover, we additionally have to consider a linear transformation – due to the system matrix A_{cl} – which is given by

$$f_{A_{\text{cl}}X}(\epsilon) = \frac{1}{|\det(A_{\text{cl}})|} f_X(A_{\text{cl}}^{-1}\epsilon). \tag{12}$$

As discussed above, the state $e(k|t)$ of the error system (9) is contained in the set Ω_k for all $k \in \mathbb{I}_{\geq 0}$; denote its distribution by $\rho_{\Omega_k}(\epsilon)$. Since $e(0|t) = 0$, we have $\rho_{\Omega_0} = \delta(\epsilon)$. By means of the above, the error can be interpreted as a random variable whose PDF is given by the recursion

$$\rho_{\Omega_{k+1}}(\epsilon) = (\rho_{A_{\text{cl}}\Omega_k} * \rho_{\mathbb{W}})(\epsilon), \tag{13}$$

for all $k \in \mathbb{I}_{\geq 1}$. With (Klenke, 2013, Theorem 1.101), we can determine that $\rho_{A_{\text{cl}}\Omega_k}(\epsilon) = 0$ if $\epsilon \notin A_{\text{cl}}\Omega_k$. Using the definition of the convolution, it follows that the support of two convolved distributions is the Minkowski sum of the two supports. This means that the support of the PDF ρ_{Ω_k} of the error at each iteration is equivalent to the associated error set at this iteration.

Remark 5. Note that if the initial state is not known (exactly), for example due to measurement noise, the recursion still holds replacing the Dirac delta function by a PDF of the initial error.

However, in the following analysis, we restrict ourselves to the assumption that $x(t)$ can be measured exactly at each time step. Considering additional measurement noise is subject to ongoing research. ■

Remark 6. In general, computing the convolutions (13) might be a challenging task, especially for higher order dimensions. On the other hand, this computation is performed offline and is a standard problem where different numerical integration schemes exist. ■

3. ROBUST ECONOMIC MPC

As introduced in the previous section, we are interested in minimizing the asymptotic average performance (6) of the system. As usual in MPC, this is approximated by solving, at each time step, a finite horizon optimal control problem.

Moreover, in this setup, we consider additive disturbances to the system. A first robust economic MPC scheme for such a setting has recently been presented by Bayer et al. (2014). There, it was shown that just transferring concepts from stabilizing robust MPC to economic MPC might not be the optimal choice with respect to the achievable asymptotic average performance. In Bayer et al. (2014), an average over the derived invariant error set (for this setup over Ω_∞) is taken into account by integrating the stage cost over the mRPI set. By doing so, all possible disturbances are considered within the optimization problem. However, no information is given about how likely these errors are when affecting the system with a bounded disturbance. Hence, this method provides a useful but conservative approximation of the closed-loop behavior of (1). In order to overcome the conservatism, it would be desirable to take the growing error sets (10) into account within the prediction, and – if possible – also the derived distribution over the error sets (13).

This can be achieved by considering the expected value over the predictions of the stage cost. This leads to the following finite horizon optimal control problem to be solved at each time instance, which we will discuss in more detail below.

$$V^*(x(t)) = \min_{\boldsymbol{c}(t)} \sum_{k=0}^{N-1} \mathbb{E}^t\big\{\ell(x(k+t), Kx(k+t) + c(k|t))\big\} + V_{\mathrm{f}}^{\mathrm{int}}(z(N|t)) \tag{14}$$

$$\text{s.t. } z(k+1|t) = A_{\mathrm{cl}} z(k|t) + Bc(k|t), \tag{15a}$$
$$z(0|t) = x(t), \tag{15b}$$
$$(z(k|t), Kz(k|t) + c(k|t)) \in \overline{\mathbb{Z}}_k \ \ \forall k \in \mathbb{I}_{[0,N-1]}, \tag{15c}$$
$$z(N|t) \in \overline{\mathbb{X}}_{\mathrm{f}}, \tag{15d}$$

where $x(k+t) = z(k|t) + \sum_{j=0}^{k-1} A_{\mathrm{cl}}^{k-j-1} w(j+t)$. The notation $\mathbb{E}^t\{\cdot\}$ is an abbreviation for the conditional probability $\mathbb{E}\{\cdot|x(t)\}$. By $\boldsymbol{c}^*(t) = \{c^*(0|t), \dots, c^*(N-1|t)\}$, we denote an arbitrary minimizer of the optimal control problem (14)–(15), which we assume to exist for simplicity.

First, we concentrate on the objective (14). We need to consider the conditional probability given $x(t)$ as this is the only known term at time t, whereas all predictions $x(k+t)$ are subject to disturbances for which we only know that $w(j+t) \in \mathbb{W}$ for all $j \in \mathbb{I}_{[0,k-1]}$ and the distribution $\rho_{\mathbb{W}}$. When evaluating the single terms, they can explicitly be written as

$$\begin{aligned} &\mathbb{E}^t\{\ell(x(t+k), Kx(t+k) + c(k|t))\} \\ &= \int_{\Omega_k} \ell(z(k|t) + \epsilon, K(z(k|t) + \epsilon) + c(k|t))\rho_{\Omega_k}(\epsilon)\mathrm{d}\epsilon \\ &=: \ell_k^{\mathrm{int}}(z(k|t), c(k|t)). \end{aligned} \tag{16}$$

In the following analysis, we will have to consider the case $k \to \infty$. Therefore, we introduce the notation

$$\ell_\infty^{\mathrm{int}}(z, c) := \limsup_{k\to\infty} \ell_k^{\mathrm{int}}(z, c). \tag{17}$$

Using (17), we can introduce the *robust optimal steady-state* (ROSS) (z_s, c_s) satisfying

$$\ell_\infty^{\mathrm{int}}(z_s, c_s) = \inf_{\substack{z = A_{\mathrm{cl}} z + c \\ (z, Kz+c) \in \overline{\mathbb{Z}}_\infty}} \ell_\infty^{\mathrm{int}}(z, c). \tag{18}$$

We assume that the ROSS is unique. If this is not the case, (z_s, c_s) denotes an arbitrary steady-state satisfying (18).

Remark 7. We conjecture that under certain assumptions on the PDF $\rho_{\mathbb{W}}$, the sequence ℓ_k^{int} converges, i.e., the limit $\lim_{k\to\infty} \ell_k^{\mathrm{int}}$ exists and (17) reduces to $\ell_\infty^{\mathrm{int}}(z, c) := \lim_{k\to\infty} \ell_k^{\mathrm{int}}(z, c)$.

For the terminal cost, we introduce the following assumption, which will in further detail be discussed in Sections 4 and 5.

Assumption 8. There exists a terminal cost $V_{\mathrm{f}}^{\mathrm{int}} : \overline{\mathbb{X}}_{\mathrm{f}} \to \mathbb{R}$ and a terminal control law $\kappa_{\mathrm{f}}(z) = Kz + c_s$ such that for all $z \in \overline{\mathbb{X}}_{\mathrm{f}}$, the following inequality holds:

$$\begin{aligned} &\mathbb{E}\big\{V_{\mathrm{f}}^{\mathrm{int}}(A_{\mathrm{cl}} z + Bc_s + A_{\mathrm{cl}}^N w)|z\big\} - V_{\mathrm{f}}^{\mathrm{int}}(z) \\ &\le -\ell_N^{\mathrm{int}}(z, c_s) + \ell_\infty^{\mathrm{int}}(z_s, c_s). \end{aligned} \tag{19}$$
■

Second, we have a closer look at the constraints (15). These constraints are taken from the tube-based robust MPC scheme presented in Chisci et al. (2001). The initial nominal state is set to the measured real state (15b), which is assumed to be known exactly. Recall that this means that the error is reset ($e(0|t) = 0$) at the beginning of each iteration. As we consider the nominal dynamics (15a), we have to tighten the pointwise-in-time constraints taking the growing error sets Ω_k into account according to (10), that is,

$$\overline{\mathbb{Z}}_k = \mathbb{Z} \ominus (\Omega_k \times K\Omega_k). \tag{20}$$

As the terminal constraint, we use

$$\overline{\mathbb{X}}_{\mathrm{f}} = O_{\max} \ominus \Omega_N, \tag{21}$$

where $O_{\max}$ is the maximal output admissible set (see, e.g., Kolmanovsky and Gilbert (1998)). Note that this terminal region is robustly invariant with respect to the disturbance $A_{\mathrm{cl}}^N \mathbb{W}$ when applying the terminal controller κ_{f}.

The proposed robust economic MPC scheme is then given as follows:

Algorithm 1 Robust Economic MPC

given: initial state x_0
  for $t = 0, 1, 2, \dots$ **do**
    solve (14)–(15)
    apply $u(t) = Kx(t) + c^*(0|t)$ to system (1)
  end for

4. ASYMPTOTIC PERFORMANCE BOUND

In this section, we investigate bounds for the asymptotic average performance (6) for the closed-loop system resulting from the application of the robust economic MPC scheme in Algorithm 1.

Theorem 9. Let Assumptions 1 and 8 be satisfied. Assume that the optimization problem (14)–(15) is feasible at time $t = 0$ for a given initial condition x_0. Then Algorithm 1 is recursively feasible and the solution of the closed-loop system

$$x(t+1) = A_{\text{cl}}x(t) + Bc^*(0|t) + w(t) \tag{22}$$

has an expected asymptotic average performance which is at least as good as that of the robust optimal steady-state, that is

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}^0\big\{\ell(x(t), Kx(t) + c^*(0|t))\big\} \le \ell_\infty^{\text{int}}(z_s, c_s). \tag{23}$$

■

Remark 10. The average performance bound $\ell_\infty^{\text{int}}(z_s, c_s)$ in (23) can be interpreted as follows: Each trajectory starting in the RPI set Ω_∞ centered at the ROSS will stay inside Ω_∞ if the terminal controller is applied, and will, for $t \to \infty$, be distributed over Ω_∞ with distribution ρ_{Ω_∞}. Hence, $\ell_\infty^{\text{int}}(z_s, c_s)$ can be interpreted as the expected average cost at the ROSS. This corresponds to the average performance bound $\ell(x^*, u^*)$ typically determined in nominal economic MPC, where (x^*, u^*) is the optimal steady-state. Concerning the left hand side in (23), the closed-loop trajectory at each time instant t is depending on previous disturbances. Thus, the result must be related to the (known) initial state $x(0) = x_0$ leading to a conditional expectation, which is a known result for example in stochastic (stabilizing) MPC (see, e.g., Cannon et al. (2009), Lorenzen et al. (2015)). ■

Proof of Theorem 9. The proof of recursive feasibility follows directly along the lines of the proof in Chisci et al. (2001). We use a candidate input given by $\tilde{\boldsymbol{c}}(t+1) = \{c^*(1|t), \dots, c^*(N-1|t), c_s\}$. The key steps are (i) the time-variant tightening of the constraints (20) and (ii) the choice of the terminal set $\overline{\mathbb{X}}_{\text{f}}$ (21) guaranteeing that $A_{\text{cl}}z + Bc_s + A_{\text{cl}}^N w \in \overline{\mathbb{X}}_{\text{f}}$ for all $z \in \overline{\mathbb{X}}_{\text{f}}$ and $w \in \mathbb{W}$.

In contrast to the robust economic MPC idea in Bayer et al. (2014), the stage cost is depending on the prediction time k. This means we can not just apply the standard average performance result as the one in Angeli et al. (2012). Here, we again make use of the candidate solution $\tilde{\boldsymbol{c}}(t+1)$. Note that due to the initial constraint (15b), the corresponding nominal candidate state sequence $\tilde{\boldsymbol{z}}(t+1)$ is not the same as $\{z^*(1|t), \dots, z^*(N|t), A_{\text{cl}}z^*(N|t) + Bc_s\}$. By $\tilde{V}(x(t), \tilde{\boldsymbol{c}}(t))$, we denote the suboptimal cost using the candidate solution $\tilde{\boldsymbol{c}}(t+1)$. This leads to

$$\begin{aligned} V^*(x(T)) - V^*(x(0)) &= \sum_{t=0}^{T-1} V^*(x(t+1)) - V^*(x(t)) \\ &\le \sum_{t=0}^{T-1} \tilde{V}(x(t+1), \tilde{\boldsymbol{c}}(t+1)) - V^*(x(t)). \end{aligned}$$

Using the conditional probability given $x(0)$, it follows with the law of iterated expectations as T is fixed that

$$\begin{aligned} &\mathbb{E}^0\big\{V^*(x(T)) - V^*(x(0))\big\} \\ &\le \mathbb{E}^0\left\{\sum_{t=0}^{T-1} \mathbb{E}\left\{\tilde{V}(x(t+1), \tilde{\boldsymbol{c}}(t+1))|x(t)\right\} - V^*(x(t))\right\}, \end{aligned} \tag{24}$$

using that all terms are bounded due to boundedness of ℓ_k^{int} on $\overline{\mathbb{Z}}_k$. For the right hand side of (24), we can compute for all $t \in \mathbb{I}_{[0,T-1]}$ for each summand

$$\begin{aligned} &\mathbb{E}\left\{\tilde{V}(x(t+1), \tilde{\boldsymbol{c}}(t+1))|x(t)\right\} - V^*(x(t)) \\ &= \sum_{k=0}^{N-1} \mathbb{E}\big\{\ell_k^{\text{int}}(z^*(k+1|t) + A_{\text{cl}}^k w(t), c^*(k+1|t))|z^*(k+1|t)\big\} \\ &\quad + \mathbb{E}\left\{V_{\text{f}}^{\text{int}}(A_{\text{cl}}z(N|t) + Bc_s + A_{\text{cl}}^N w(t))|z^*(N|t)\right\} \\ &\quad - \sum_{k=0}^{N-1} \ell_k^{\text{int}}(z^*(k|t), c^*(k|t)) - V_{\text{f}}^{\text{int}}(z^*(N|t)), \end{aligned} \tag{25}$$

where we have used that the nominal candidate sequence $\tilde{\boldsymbol{z}}(t+1)$ is given by applying $\tilde{\boldsymbol{c}}(t+1)$ to the nominal system (7) starting at $x(t+1) = z^*(1|t) + w(t)$. With the definition of ℓ_k^{int} in (16), it follows that

$$\begin{aligned} &\mathbb{E}\{\ell_k^{\text{int}}(z^*(k+1|t) + A_{\text{cl}}^k w, c^*(k+1|t))|z^*(k+1|t)\} \\ &= \ell_{k+1}^{\text{int}}(z^*(k+1|t), c^*(k+1|t)), \end{aligned}$$

and thus, it follows from (25) that

$$\begin{aligned} &\mathbb{E}\left\{\tilde{V}(x(t+1), \tilde{\boldsymbol{c}}(t+1))|x(t)\right\} - V^*(x(t)) \\ &= \ell_N^{\text{int}}(z^*(N|t), c_s) - \ell\big(x(t), Kx(t) + c^*(0|t)\big) - V_{\text{f}}^{\text{int}}(z^*(N|t)) \\ &\quad + \mathbb{E}\left\{V_{\text{f}}^{\text{int}}(A_{\text{cl}}z(N|t) + Bc_s + A_{\text{cl}}^N w(t))|z(N|t)\right\}. \end{aligned} \tag{26}$$

Using Assumption 8, (24), and (26), we can derive that

$$\begin{aligned} &\mathbb{E}^0\big\{V^*(x(T)) - V^*(x(0))\big\} \\ &\le \mathbb{E}^0\left\{\sum_{t=0}^{T-1} \big(\ell_\infty^{\text{int}}(z_s, c_s) - \ell(x(t), Kx(t) + c^*(0|t))\big)\right\} \end{aligned} \tag{27}$$

By dividing both sides with T, we can see that

$$\begin{aligned} &\frac{1}{T}\big(\mathbb{E}^0\{V^*(x(T))\} - V^*(x(0))\big) \\ &\le \ell_\infty^{\text{int}}(z_s, c_s) - \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}^0\big\{\ell(x(t), Kx(t) + c^*(0|t))\big\}, \end{aligned}$$

where we have used that $\mathbb{E}^0\{V^*(x(0))\} = V^*(x(0))$ as well as $\mathbb{E}^0\{\ell_\infty^{\text{int}}(z_s, c_s)\} = \ell_\infty^{\text{int}}(z_s, c_s)$. As the terms on the left hand side are finite and by taking the limit inferior as $T \to \infty$, it follows that the left hand side vanishes, and thus,

$$\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}^0\big\{\ell(x(t), Kx(t) + c^*(0|t))\big\} \le \ell_\infty^{\text{int}}(z_s, c_s),$$

which proves the average performance bound (23). □

5. QUADRATIC APPROXIMATION OF THE TERMINAL COST

In the previous section, we have derived a statement bounding the closed-loop asymptotic average performance. This result is based on Assumption 8 stating a condition for the terminal cost $V_{\text{f}}^{\text{int}}$. One can show that by choosing the terminal cost such that

$$\begin{aligned} V_{\text{f}}^{\text{int}}(z(N|t)) = \sum_{k=N}^{\infty} &\big(\mathbb{E}^t\{\ell(x(k+t), Kx(k+t) + c_s)\} \\ &- \ell_\infty^{\text{int}}(z_s, c_s)\big), \end{aligned} \tag{28}$$

condition (19) in Assumption 8 is satisfied with equality. However, finding the terminal cost satisfying (28) or another appropriate terminal cost in accordance to Assumption 8 might be difficult. In order to overcome this difficulty, we briefly present an approach to determine a quadratic approximation for the terminal cost. Yet, we have to point out that using this quadratic approximation will slightly deteriorate the a priori determinable average performance bound (23), which will be shown in the following. The idea of the concept is derived from Amrit et al. (2011), where a quadratic terminal cost approximation is determined in the nominal case, and we use the same assumption on the cost function.

Assumption 11. The cost function ℓ is twice continuously differentiable. ∎

Due to limited space, a detailed analysis and discussion of this approach is omitted and will be presented in a subsequent publication.

Using the quadratic approximation idea from Amrit et al. (2011), a terminal cost of the form

$$V_{\mathrm{f}}^{\mathrm{int}}(z(N|t)) = z(N|t)^T P z(N|t) + p^T z(N|t) \tag{29}$$

can be derived. The linear coefficient vector p is given according to $p^T = q^T(I - A_{\mathrm{cl}})^{-1}$, with q being the gradient $\ell_z(z, \kappa_{\mathrm{f}}(z))$ evaluated at z_s. The matrix P is the solution of the Lyapunov equation $P = A_{\mathrm{cl}}^T P A_{\mathrm{cl}} + Q^*$, where $Q^* > 0$ is an over-approximation for the Hessian $\ell_{zz}(z, \kappa_{\mathrm{f}}(z))$ for all $z \in \overline{\mathbb{X}}_{\mathrm{f}} \oplus \Omega_\infty$. For $V_{\mathrm{f}}^{\mathrm{int}}$ as defined in (29), Assumption 8 is not necessarily satisfied, but instead the inequality

$$\begin{aligned}&\mathbb{E}\big\{V_{\mathrm{f}}^{\mathrm{int}}(A_{\mathrm{cl}}z + Bc_s + A_{\mathrm{cl}}^N w)|z\big\} - V_{\mathrm{f}}^{\mathrm{int}}(z)\\ &\leq -\ell_N^{\mathrm{int}}(z, c_s) + \ell(z_s, Kz_s + c_s) + \frac{1}{2}\int_{\mathbb{W}} \epsilon^T P \epsilon \rho_{\mathbb{W}}(\epsilon)\mathrm{d}\epsilon.\end{aligned}$$

From here, using (29), we can follow the steps of the proof of Theorem 9, which leads to a performance bound different to the one in (23), namely

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathbb{E}^0\big\{\ell(x(t), Kx(t) + c^*(0|t))\big\}\\ &\qquad\leq \ell(z_s, Kz_s + c_s) + \frac{1}{2}\int_{\mathbb{W}} \epsilon^T P \epsilon \rho_{\mathbb{W}}(\epsilon)\mathrm{d}\epsilon.\end{aligned} \tag{30}$$

Having a closer look at the statements in (23) and (30), one can show that

$$\begin{aligned}\ell_\infty^{\mathrm{int}}(z_s, c_s) &= \int_{\Omega_\infty} \ell(z_s + \epsilon, K(z_s + \epsilon) + c_s)\rho_{\Omega_\infty}(\epsilon)\mathrm{d}\epsilon\\ &\leq \ell(z_s, Kz_s + c_s) + \frac{1}{2}\int_{\mathbb{W}} \omega^T P \omega \rho_{\mathbb{W}}(\omega)\mathrm{d}\omega,\end{aligned} \tag{31}$$

and furthermore, the performance bound in (30) is a quadratic approximation of the bound in (23). Hence, using a quadratic approximation of the terminal cost leads to a degradation of the performance bound, namely to a quadratic approximation of the performance bound (23).

Remark 12. If a quadratic stage cost function of the form $\ell(x, u) = x^T Q x + u^T R u$ with $Q \geq 0$ and $R > 0$ is considered, the statement in (31) is satisfied with equality, and this bound corresponds to the result in stochastic stabilizing MPC (see, e.g., Cannon et al. (2009), Lorenzen et al. (2015)).

6. NUMERICAL EXAMPLE

In the following, we apply the presented approach to an example from literature, namely to the linearized model of a continuous stirred tank reactor (CSTR) presented in Pannocchia and Kerrigan (2005), which also has been used in the context of robust economic MPC in Bayer et al. (2014). Those results are compared to the ones achieved with the current approach.

The reaction taking place in the reactor is a single exothermic irreversible first-order reaction of the form $A \xrightarrow{k} B$. The considered states are the concentration of A, c^A, and the temperature T in the reactor. As an input, the wall temperature T^c can be used. The model is linearized around the state $(c_s^A, T_s) = (0.5\,\frac{\mathrm{mol}}{\mathrm{L}}, 350\,\mathrm{K})$ with $T_s^c = 300\,\mathrm{K}$. Using the same parameters as in Pannocchia and Kerrigan (2005) leads to the dynamics

$$\begin{aligned}x(t+1) = &\begin{bmatrix} 0.7776 & -0.0045 \\ 26.6185 & 1.8555 \end{bmatrix} x(t) + \begin{bmatrix} -0.0004 \\ 0.2907 \end{bmatrix} u(t)\\ &+ \begin{bmatrix} -0.0002 & 0.0893 \\ 0.1390 & 1.2267 \end{bmatrix} w(t).\end{aligned}$$

For the constraints, we assume $\mathbb{X} = \{x \in \mathbb{R}^2 : |x_1| \leq 0.5, |x_2| \leq 5\}$, $\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 15\}$, and the disturbance is uniformly distributed on the set $\mathbb{W} = \{w \in \mathbb{R}^2 : |w_1| \leq 2, |w_2| \leq 0.1\}$. For the pre-stabilization, we use a feedback $K = [-0.6457, -5.4157]$.

For the stage cost function, we assume two objectives. The first goal is to minimize the concentration of A, thus, maximize the concentration of the desired product B. Moreover, the temperature should be kept in the interval $x_2 \in [-2, 2]$ via soft constraints (in contrast to the hard constraints $|x_2| \leq 5$). Thus, the cost function is given by

$$\ell(x, u) = x_1 + \begin{cases} 10x_2^2 + 40x_2 + 40 & \text{for } x_2 < -2 \\ 0 & \text{for } -2 \leq x_2 < 2 \\ 10x_2^2 - 40x_2 + 40 & \text{for } 2 \leq x_2 \end{cases}.$$

We employ the presented algorithm with $N = 20$. For the terminal cost, we use a quadratic approximation following the idea in Section 5 with $Q^* = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 5.988 \end{smallmatrix}\right]$ and $q^T = (1, 0)$. This leads to $P = \left[\begin{smallmatrix} 18067.43 & 61.52 \\ 61.52 & 6.50 \end{smallmatrix}\right]$ and $p^T = (4.5455, 0)$.

We can compute the ROSS to be $z_s = (-0.0239, 1.3339)^T$ with the associated input $c_s = 5.4679$. Computing the bounds on the asymptotic average performance, we can see that $\ell_\infty^{\mathrm{int}}(z_s, c_s) = -0.024301$, whereas for the quadratic approximation, we receive $\ell(z_s, Kz_s + c_s) + \frac{1}{2}\int_{\mathbb{W}} \epsilon^T P \epsilon \rho_{\mathbb{W}}(\epsilon)\mathrm{d}\epsilon = 0.336991$. We can see that the difference between the two bounds is quite large for this example, which is caused by the rather large terminal region and the inevitable symmetry of the terminal cost. Due to the quadratic approximation of the terminal cost, the predictable a priori asymptotic average performance bound (30) is quite conservative, as shown in the following.

We now compare the closed-loop performance of this approach to the results derived for previous concepts of robust economic MPC. In the first approach, the disturbance is not considered in the stage cost, leading to

$$\ell^{(i)}(z, c) = \ell(z, Kz + c).$$

This is an approach resembling the standard idea in tube-based robust MPC. The second approach, presented in Bayer et al. (2014), considers the influence of the disturbance on the performance by using the modified stage cost

$$\ell^{(ii)}(z, c) = \int_{\{z\}\oplus\Omega_\infty} \ell(x, Kx + c)\mathrm{d}x.$$

In this approach, as already discussed in the introduction and in Section 3, no distribution of the error is assumed to be known, but an average of ℓ over Ω_∞ is used as a stage cost. Using these two approaches, we can compute their associated optimal steady states at $z_s^{(i)} = (-0.0358, 2.0009)^T$ and $z_s^{(ii)} = (-0.0076, 0.4275)^T$, respectively.

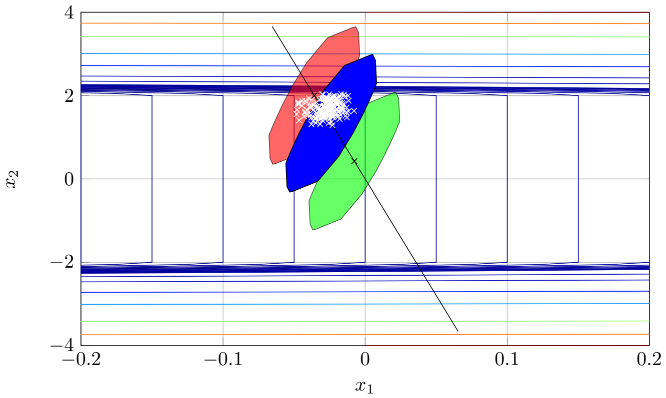

Fig. 1. Contour plot of the stage cost function. The sets represent $\{z_s\} \oplus \Omega_\infty$ (blue), $\{z_s^{(i)}\} \oplus \Omega_\infty$ (red), and $\{z_s^{(ii)}\} \oplus \Omega_\infty$ (green), respectively. The black line represents all feasible nominal steady-states, the white crosses are one closed-loop sequence of real states determined with Algorithm 1 (for 200 iterations).

Considering the asymptotic average performance for the closed-loop system, we receive (averaged over 20 simulations):

Presented reMPC	Standard MPC ($\ell^{(i)}$)	reMPC ($\ell^{(ii)}$)
–0.0298	+ 0.3410	–0.0076

We can see that taking the additional stochastic information into account within the economic MPC scheme significantly improves the average performance. Even though the previously presented approach based on $\ell^{(ii)}$ outperforms the standard MPC approach based on $\ell^{(i)}$, considering stochastic information provides the best asymptotic average performance.

When comparing the steady-state behavior in Figure 1, we can see that for the MPC approach based on $\ell^{(i)}$ (red), the disturbances might push the system into "expensive" areas. This is the case because no disturbance is considered within the stage cost. The robust economic MPC based on $\ell^{(ii)}$ (green) keeps the system for all possible disturbances within the desired temperature range. In contrast to these, our newly presented approach takes into consideration that errors at the edge of the mRPI set are unlikely to occur. Thus, the optimal steady-state is closer to the edge of the desired temperature range (blue). Comparing the closed loop, one can see that while for the two other approaches the nominal system is best kept at the optimal steady-state (see Bayer et al. (2014) for a detailed discussion), in the presented approach the nominal system starts at the states of the real system (due to (15b)) (white crosses). Thus, the closed-loop can – depending on the disturbances - even be driven closer to the edge of the desired temperature range, and hence, the average performance is improved. Moreover, the achieved asymptotic average performance is far better than the determined (conservative) performance bound.

7. CONCLUSION

A new approach for robust economic MPC was presented taking stochastic information into account in order to improve the closed-loop performance. The key idea is to explicitly calculate the expected value cost of the open-loop predictions within the optimization problem. We were able to provide bounds on the closed-loop average performance which are based on an appropriate terminal cost. As finding such an appropriate terminal cost can be difficult, we briefly presented an approach for determining a quadratic approximation. However, as also seen in the example, this might deteriorate the a priori determinable performance bound.

Further possibilities for finding a less conservative approximation for the terminal cost are currently investigated. In addition, probabilistic constraints could be considered in this framework. Finally, further research is needed on how the performance is influenced by the chosen feedback used in the predictions.

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