Distributed Model Predictive Control - Recursive feasibility under inexact dual optimization ⋆

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Abstract

We propose a novel model predictive control (MPC) formulation, that ensures recursive feasibility, stability and performance under inexact dual optimization. Dual optimization algorithms offer a scalable solution and can thus be applied to large distributed systems. Due to constraints on communication or limited computational power, most real-time applications of MPC have to deal with inexact minimization. We propose a modified optimization problem inspired by robust MPC which offers theoretical guarantees despite inexact dual minimization. The approach is not tied to any particular optimization algorithm, but assumes that the feasible optimization problem can be solved with a bounded suboptimality and constraint violation. In combination with a distributed dual gradient method, we obtain a priori upper bounds on the number of required online iterations. The design and practicality of this method are demonstrated with a benchmark numerical example.

1 Introduction

Model predictive control (MPC) is a well-established control method, that can be used to control complex dynamical systems and guarantee constraint satisfaction [34]. One of the main limitations to control a system with MPC comes from computational issues, since in each time step an optimization problem has to be solved. In order to apply MPC to large-scale systems, we have to consider distributed approaches, which fall in the domain of distributed MPC (DMPC) [27,28]. If we want to facilitate DMPC applications to fast (physically) interconnected networks, we typically need scalable distributed optimization algorithms with bounds on the number of required iterations.

Dual optimization algorithms such as the alternating direction method of multipliers (ADMM), dual gradient methods and proximal decomposition have been studied to solve DMPC optimization problems online [17,31,30]. While these algorithms enable a fully distributed implementation and asymptotically converge to the optimal central solution, real-time requirements lead to early termination and an inexact solution. Contrary to primal decomposition methods [37], these inexact solutions based on dual optimization do not necessarily satisfy the posed constraints (dynamic, state and input constraints) in the MPC optimization problem. This necessitates additional modifications to ensure recursive feasibility and stability of the resulting MPC scheme.

Related work

In [9] DMPC without terminal constraints is investigated and a sufficient stopping condition for the distributed iteration based on a candidate solution is presented. For this approach no prior bound on the number of required iterations can be given.

In [19] a primal optimization algorithm with constraint violations in the dynamic equality constraints is investigated. Recursive feasibility is ensured with an appropriate state and input constraint tightening.

In [35,29] constraint violations in the inequality constraints due to inexact dual optimization are addressed with an appropriate (constant or adaptive) constraint tightening. Constraint violations in the posed dynamic equality constraints are avoided by using a condensed formulation [29] or projecting the intermediate solution to the set of dynamically feasible trajectories [35]. Both approaches are, however, unsuited for distributed large-scale systems.

In [7] constraint violations in inequality constraints and dynamic equality constraints are considered by using

⋆ This paper was not presented at any IFAC meeting.

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an appropriate constraint tightening. Recursive feasibility is ensured by choosing the tolerance and thus the constraint tightening adaptively. As a consequence, the number of iterations can vary and global communication is required to enable this adaptation. In [6] a similar constraint tightening is used for a distributed hierarchical MPC scheme.

Contribution

We propose a new framework to ensure recursive feasibility of inexact DMPC resulting from finite dual iterations. This consists of a constant constraint tightening and a stabilizing controller, motivated by robust MPC [2]. To avoid an overly conservative constraint tightening, we propose a modified optimization problem and employ a different candidate solution, that explicitly takes the inexactness into account. This presents a general procedure which is applicable to different MPC setups. By combining this framework with a dual distributed gradient algorithm, we obtain an a priori upper bound for the number of dual iterations to ensure recursive feasibility. Compared to [29,7,9], no adaptive constraint tightening is required. Furthermore, compared to [19,35,7,6], no centralized operations are necessary, thus allowing a fully distributed implementation for large-scale systems.

Outline

The remainder of this paper is structured as follows: Section 2 presents the nominal distributed MPC formulation and explains the problem inherent in inexact dual optimization. Section 3 presents the modified formulation, derives closed-loop properties under inexact minimization and presents a corresponding distributed dual iteration scheme. Section 4 illustrates the practicality and simplicity of the proposed framework with a numerical example. Section 5 concludes the paper.

In the appendix, these results are extended to MPC without terminal ingredients, unreachable setpoints, multi-step MPC and the distributed offline computation of the terminal ingredients is detailed.

The main content (without appendix) is accepted to be published in Automatica as a brief paper .

2 Distributed Model Predictive Control

Notation

The real numbers are $\mathbb{R}$, the positive real numbers are $\mathbb{R}_{>0} = \{r \in \mathbb{R} | r > 0\}$ and the natural numbers are $\mathbb{N}$. Given vectors $a_i \in \mathbb{R}^{n_i}$, we abbreviate the column vector $[a_1^\top, \ldots, a_n^\top]^\top = (a_1, \ldots, a_n)$. The quadratic norm with respect to a positive definite matrix $Q = Q^\top$ is denoted by $\|x\|_Q^2 = x^\top Q x$ and the minimal and maximal eigenvalue of Q are denoted by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$, respectively. For a polytopic constraint $Ay \leq b$, we define an ϵ-feasible solution as any vector y that satisfies $Ay \leq b + \epsilon \mathbf{1}$, with $\epsilon \in \mathbb{R}_{>0}$ and the vector of ones $\mathbf{1} = [1, \ldots, 1]^\top$. We call a vector ϵ-strictly feasible if it satisfies $Ay \leq b - \epsilon \mathbf{1}$. The Minkowski sum of two sets $S, T \subset \mathbb{R}^n$ is denoted by

$$S \oplus T = \{x | \exists s \in S, t \in T : x = s + t\}.$$

A distributed system is represented as a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with nodes $\mathcal{N}$ and edges $\mathcal{E}$. Each node $i \in \mathcal{N}$ corresponds to a subsystem with local state $x_i \in \mathbb{R}^{n_i}$ and local input $u_i \in \mathbb{R}^{m_i}$. The neighborhood of a subsystem i is given by $\mathcal{N}_i = \{j | (i,j) \in \mathcal{E}\} \cup \{i\}$, with $x_{\mathcal{N}_i} \in \mathbb{R}^{n_{\mathcal{N}_i}}$, $n_{\mathcal{N}_i} = \sum_{j \in \mathcal{N}_i} n_j$.

2.1 Problem setup

The distributed linear discrete-time [1] system is given by

$$x_i(t+1) = A_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + B_i u_i(t), \quad i \in \mathcal{N}, \tag{1}$$

with polytopic state and input constraints of the form

$$x_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i} = \{x_{\mathcal{N}_i} | H_{\mathcal{N}_i} x_{\mathcal{N}_i} \leq h_{\mathcal{N}_i}\}, \tag{2}$$
$$u_i \in \mathcal{U}_i = \{u_i | L_i u_i \leq l_i\}, \tag{3}$$

where $h_{\mathcal{N}_i} \in \mathbb{R}^{p_i}_{>0}$ and $l_i \in \mathbb{R}^{q_i}_{>0}$. We consider the general case, where the control input is given by

$$u_i(t) = K_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + v_i(t), \tag{4}$$

where K is some existing distributed controller and v is the input calculated using distributed MPC. If no such feedback is known, we can always set $K = 0$. However, including this feedback can reduce the conservatism and mitigate the deteriorating effects of suboptimality on closed-loop stability. The overall system is given by

$$x(t+1) = Ax(t) + Bu(t) = \underbrace{(A + BK)}_{=:A_K} x(t) + Bv(t), \tag{5}$$

with the polytopic constraints

$$\mathcal{X} = \{x | x_{\mathcal{N}_i} \in \mathcal{X}_{\mathcal{N}_i} \forall i \in \mathcal{N}\} = \{x | Hx \leq h\} \subseteq \mathbb{R}^n,$$
$$\mathcal{U} = \mathcal{U}_1 \times \ldots \mathcal{U}_{|\mathcal{N}|} = \{u | Lu \leq l\} \subseteq \mathbb{R}^m,$$

where $l \in \mathbb{R}^q_{>0}$ and $h \in \mathbb{R}^p_{>0}$. We consider a structured quadratic stage cost $\ell(x, v) = \|x\|_Q^2 + \|v\|_R^2$, with block

[1] In case of stiff continous-time dynamics, an implicit discretization method is typically used, yielding the discrete-time model $F_{\mathcal{N}_i} x_{\mathcal{N}_i}(t+1) = A_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + B_i u_i(t)$. The following derivations are equally applicable for such models and preserve the distributed structure, compare [24].

diagonal positive definite matrices Q and R. We consider an MPC framework including a terminal cost and terminal set. To this end, we make the following assumption.

Assumption 1 *There exists a terminal cost* $V_f(x) = \sum_{i\in\mathcal{N}} \|x_i\|_{P_i}^2 = \|x\|_P^2$ *with a block diagonal matrix* P*, a distributed terminal controller* $u_i = K_{f,\mathcal{N}_i} x_{\mathcal{N}_i}$*, and a distributed compact polytopic set* $\mathcal{X}_f = \{x | Fx \leq f\}$*, such that the following conditions hold for each* $x_f \in \mathcal{X}_f$

$$V_f((A+BK_f)x_f) \leq V_f(x) - \ell(x_f, (K_f - K)x_f), \tag{6a}$$

$$x_f \in \mathcal{X}, \quad u_f = K_f x_f \in \mathcal{U}, \tag{6b}$$

$$(A+BK_f)x_f \in \mathcal{X}_f. \tag{6c}$$

Remark 2 *In [3] distributed linear matrix inequalities (LMIs) are presented that can be used to compute a distributed terminal cost and an ellipsoidal terminal set* $\mathcal{X}_f$*. Ellipsoidal terminal constraints lead to a (distributed) quadratically constrained quadratic program (QCQP), which makes the online optimization more complex. Methods to obtain a distributed polytopic terminal set* $\mathcal{X}_f$ *are for example given in [17,38]. The offline computation of the distributed terminal ingredients is discussed in more detail in Appendix A.1. The proposed framework can also be used without such terminal ingredients, which is discussed in Appendix A.2,A.3.*

The open-loop cost of a state sequence $x(\cdot|t) \in \mathbb{R}^{n\times N+1}$ and an input sequence $v(\cdot|t) \in \mathbb{R}^{m\times N}$ with the prediction horizon $N \in \mathbb{N}$ is defined as

$$J_N(x(\cdot|t), v(\cdot|t)) := \sum_{k=0}^{N-1} \ell(x(k|t), v(k|t)) + V_f(x(N|t)).$$

The standard MPC optimization problem is given by

$$\begin{aligned} V_N(x(t)) = \min_{v(\cdot|t),x(\cdot|t)} \ & J_N(x(\cdot|t), v(\cdot|t)) \\ \text{s.t. } & x(k+1|t) = A_K x(k|t) + Bv(k|t), \\ & x(0|t) = x(t), \quad x(N|t) \in \mathcal{X}_f, \\ & x(k|t) \in \mathcal{X}, \quad u(k|t) = v(k|t) + Kx(k|t) \in \mathcal{U}. \end{aligned} \tag{7}$$

The solution to this optimization problem is the value function V_N and optimal state and input trajectories $(x^*(\cdot|t),\ v^*(\cdot|t))$ that satisfy the dynamic equality constraint and the state and input constraints. Problem (7) is a distributed quadratic program, the solution of which is discussed in Sections 2.2, 3.5.

For the closed-loop operation the first step of the optimal input $v^*(\cdot|t)$ is applied to the system (5), resulting in the following closed-loop system dynamics:

$$x(t+1) = A_K x(t) + v^*(0|t) = x^*(1|t). \tag{8}$$

The following theorem is a standard result in MPC and establishes the desired properties.

Theorem 3 *[34] Let Assumption 1 hold and assume that Problem* (7) *is feasible at* $t = 0$*. Then Problem* (7) *is recursively feasible and the origin* $x = 0$ *is asymptotically stable for the resulting closed-loop system* (8)*.*

2.2 Distributed (dual) optimization

In the following, we motivate why we consider inexact dual optimization and explain why it necessitates modifications to Problem (7). Most theoretical results for MPC (such as Thm. 3) assume that the optimal solution to (7) is obtained in real time, which is rarely achievable in practice.

If primal optimization methods are used, Theorem 3 remains valid with inexact optimization assuming a suitable initialization [37,36]. However, an application of primal optimization methods to large-scale distributed systems suffers from various difficulties, including initialization and scalability.

Thus, we consider dual optimization algorithms [17,31,30], which only require *neighbor-to-neighbor* communication and can be implemented in a fully distributed manner. The main drawback of dual optimization is that the constraints (dynamic, state and input) are not necessarily satisfied after finite iterations. This necessitates additional modifications to enable theoretical guarantees after finite iterations, compare [35,7]. In the following, we provide a novel MPC formulation which is suitable for distributed computation and explicitly takes the inexact dynamics of approximate solutions into account.

3 Inexact Distributed MPC

In the following, we consider bounds on the accuracy ϵ, interpret them as disturbances and use tools from robust MPC [2] to compensate the effects of inexact minimization. The proposed modifications are inspired by [7] and directly take the inexactness of the solver into account. By making use of an inexact candidate solution, we obtain a formulation that requires no adaptation and thus no global communication.

3.1 Inexact MPC and constraint tightening

Define an accuracy for the dynamic, state, input and terminal constraints and strict feasibility ϵ_z, ϵ_x, ϵ_u, ϵ_f, $\epsilon_\lambda \in \mathbb{R}_{>0}$, given by the user. Consider relaxation parameters

$$\epsilon_{z,k} := \epsilon_\lambda + (N-1-k)(\epsilon_\lambda + \epsilon_z), \quad k = 0, \ldots, N-1, \tag{9}$$

and the sets $\mathcal{W}_k = \{w \in \mathbb{R}^n | \|w\|_\infty \leq \epsilon_{z,k} + \epsilon_z\}$. We tighten the constraints using the k-step support function [4], which for some $a \in \mathbb{R}^n$ and $k \in \mathbb{N}$ is defined as

$$\sigma_{\mathcal{W}}(a,k) = \sup_{w \in \mathcal{W}_0^k} a^\top y(k), \tag{10}$$
$$\text{s.t. } y(l+1) = A_K y(l) + w(l), \quad y(0) = 0.$$

The tightened state and input constraints are given by

$$\overline{\mathcal{X}}_k = \{x | Hx \leq \overline{h}_k\}, \quad \overline{\mathcal{U}}_k = \{u | Lu \leq \overline{l}_k\},$$

with

$$\overline{h}_{j,k} = h_j - \sigma_{\mathcal{W}}(H_j^\top, k) - \epsilon_x - k(\epsilon_x + \epsilon_\lambda), \tag{11}$$
$$\overline{l}_{j,k} = l_j - \sigma_{\mathcal{W}}(K^\top L_j^\top, k) - \epsilon_u - k(\epsilon_u + \epsilon_\lambda). \tag{12}$$

Here, $\overline{h}_{j,k}$ denotes the j-th component of $\overline{h}_k$, h_j the j-th component of h and H_j the j-th row of H, $j \leq p$. The evaluation of the k-step support function amounts to solving a distributed linear program (LP) offline. The resulting tightened constraints preserve the distributed structure and can equally be represented with the local polytopic sets $\overline{\mathcal{X}}_{\mathcal{N}_i,k}, \overline{\mathcal{U}}_{i,k}$.

Assumption 4 *Consider the terminal cost and controller from Assumption 1. There exists a compact tightened terminal set* $\overline{\mathcal{X}}_f = \{x | Fx \leq \overline{f}\}$*, such that the following conditions hold*

$$\overline{\mathcal{X}}_{f,\epsilon} \bigoplus_{k=0}^{N-1} A_K^{N-1-k} \mathcal{W}_k \subseteq \mathcal{X}_f, \tag{13a}$$
$$\overline{\mathcal{X}}_{f,\epsilon} \oplus A_K^{N-1} \mathcal{W}_0 \subseteq \{x | Hx \leq \overline{h}_{N-1} - \mathbf{1}_p \epsilon_\lambda\}, \tag{13b}$$
$$K_f(\overline{\mathcal{X}}_{f,\epsilon} \oplus A_K^{N-1} \mathcal{W}_0) \subseteq \{u | Lu \leq \overline{l}_{N-1} - \mathbf{1}_q \epsilon_\lambda\}, \tag{13c}$$
$$(A + BK_f)\left(\overline{\mathcal{X}}_{f,\epsilon} \oplus A_K^{N-1} \mathcal{W}_0\right) \subseteq \overline{\mathcal{X}}_{f,\lambda}, \tag{13d}$$
$$\overline{\mathcal{X}}_{f,\epsilon} := \{x | Fx \leq \overline{f} + \mathbf{1}_r \epsilon_f\}, \; \overline{\mathcal{X}}_{f,\lambda} := \{x | Fx \leq \overline{f} - \mathbf{1}_r \epsilon_\lambda\}.$$

The sets $\overline{\mathcal{X}}_{f,\epsilon}$, $\overline{\mathcal{X}}_{f,\lambda}$ are needed to study strict recursive feasibility (ϵ_λ) under inexact minimization (ϵ_f). A sufficient condition for (13b) is $\overline{\mathcal{X}}_{f,\epsilon} \subseteq \overline{\mathcal{X}}_N$. In case $K_f = K$, $K_f \overline{\mathcal{X}}_{f,\epsilon} \subseteq \overline{\mathcal{U}}_N$ is a sufficient condition for (13c). Condition (13d) requires contractivity of the terminal set, despite the additive disturbance w_0.

If the terminal set in Assumption 1 is contractive, Assumption 4 can be satisfied with the following design procedure: for a fixed accuracy ϵ and prediction horizon N, compute the tightened constraints (11). Then scale the terminal set $\mathcal{X}_f$ such that conditions (13a)-(13c) are satisfied. Finally, verify that condition (13d) is satisfied. If this is not the case, decrease ϵ and start over. In the appendix, we show that the proposed framework can also be used without constructing a terminal set.

With this, we define the modified optimization problem

$$\min_{v(\cdot|t), z(\cdot|t)} \; J_N(z(\cdot|t), v(\cdot|t)) \tag{14a}$$
$$\text{s.t. } \|A_K z(k|t) + Bv(k|t) - z(k+1|t)\|_\infty \leq \epsilon_{z,k}, \tag{14b}$$
$$z(k|t) \in \overline{\mathcal{X}}_k, \quad v(k|t) + Kz(k|t) \in \overline{\mathcal{U}}_k, \tag{14c}$$
$$k = 0, \dots, N-1,$$
$$z(N|t) \in \overline{\mathcal{X}}_f, \tag{14d}$$
$$z(0|t) = x(t). \tag{14e}$$

Compared to the original optimization Problem (7), the state and input constraints are tightened and the dynamic equality constraints are relaxed to inequality constraints. We do not try to find a solution that exactly satisfies the dynamic constraints, but only consider a relaxed dynamic constraint with the parameter $\epsilon_{z,k}$. This relaxation will allow us to construct a feasible candidate solution which again does not exactly satisfy the dynamic constraints. This is the key insight and novelty in order to prove recursive feasibility and stability under inexact minimization. The resulting Problem (14) is a distributed quadratic program with linear inequality constraints.

To study recursive feasibility of (14) under the inexact DMPC we introduce the notion of ϵ-feasible solutions.

Definition 5 *An ϵ-feasible solution to* (14) *is any pair* $(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t))$*, that satisfies*

$$\|A_K z_\epsilon(k|t) + Bv_\epsilon(k|t) - z_\epsilon(k+1|t)\|_\infty \leq \epsilon_{z,k} + \epsilon_z, \tag{15}$$
$$Hz_\epsilon(k|t) \leq \overline{h}_k + \mathbf{1}_p \epsilon_x,$$
$$L(v_\epsilon(k|t) + Kz_\epsilon(k|t)) \leq \overline{l}_k + \mathbf{1}_q \epsilon_u,$$
$$Fz_\epsilon(N|t) \leq \overline{f} + \mathbf{1}_r \epsilon_f, \quad z_\epsilon(0|t) = x(t).$$

This formulation allows a constraint violation for the posed constraints (14b)-(14d) by ϵ_z, ϵ_x, ϵ_u, and ϵ_f, respectively. A corresponding algorithm to ensure an ϵ-feasible solution with finite iterations is presented in Section 3.5.

3.2 Feasible consolidated trajectory

In order to characterize the feasibility of on an ϵ-feasible solution, we consider the consolidated [2] trajectory [7].

Proposition 6 *Let Assumptions 1 and 4 hold. Given an ϵ-feasible solution* (15) $z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)$ *at time t, the*

[2] The feasibility recovery scheme described in [19] to obtain a (dynamically) feasible solution is comparable to the definition of the consolidated trajectory.

consolidated state and input trajectories $\overline{x}_\epsilon(\cdot|t)$, $\overline{u}_\epsilon(\cdot|t)$

$$\begin{aligned}\overline{x}_\epsilon(k+1|t) &:= A_K\overline{x}_\epsilon(k|t) + Bv_\epsilon(k|t), \quad \overline{x}_\epsilon(0|t) := x(t),\\ \overline{u}_\epsilon(k|t) &:= K\overline{x}_\epsilon(k|t) + v_\epsilon(k|t), \end{aligned} \tag{16}$$

satisfy

$$\overline{x}_\epsilon(k|t) \in \mathcal{X}, \quad \overline{u}_\epsilon(k|t) = v_\epsilon(k|t) + K\overline{x}_\epsilon(k|t) \in \mathcal{U},$$
$$\overline{x}_\epsilon(N|t) \in \mathcal{X}_f.$$

PROOF. The inexact relaxed dynamic constraint (15) can be equivalently written as a dynamic equality constraint with an additive disturbance

$$z_\epsilon(k+1|t) = A_K z_\epsilon(k|t) + Bv_\epsilon(k|t) + w_k, w_k \in \mathcal{W}_k. \tag{17}$$

The consolidated trajectory (16) satisfies

$$\overline{x}_\epsilon(k|t) \in \{z_\epsilon(k|t)\}\bigoplus_{l=0}^{k-1} A_K^{k-l-1}\mathcal{W}_l \overset{(9)}{\subseteq} \{z_\epsilon(k|t)\}\bigoplus_{l=0}^{k-1} A_K^{l}\mathcal{W}_0, \tag{18}$$

which implies

$$\begin{aligned} H_j\overline{x}_\epsilon(k|t) &\overset{(10)}{\leq} H_j z_\epsilon(k|t) + \sigma_{\mathcal{W}}(H_j^\top, k)\\ &\overset{\text{Def. 5}}{\leq} \overline{h}_{j,k} + \epsilon_x + \sigma_{\mathcal{W}}(H_j^\top, k) \overset{(11)}{\leq} h_j,\\ L_j\overline{u}_\epsilon(k|t) &\overset{(10)}{\leq} L_j(v_\epsilon(k|t) + Kz_\epsilon(k|t)) + \sigma_{\mathcal{W}}(K^\top L_j^\top, k)\\ &\overset{\text{Def. 5}}{\leq} \overline{l}_{j,k} + \epsilon_u + \sigma_{\mathcal{W}}(K^\top L_j^\top, k) \overset{(12)}{\leq} l_j. \end{aligned}$$

Terminal constraint satisfaction follows by condition (13a) in combination with the characterization (18) for $k = N$. □

Proposition 6 shows that the consolidated trajectory based on the inexact optimization has all the desirable properties of the standard optimal solution $x^*(\cdot|t)$, $u^*(\cdot|t)$ to Problem (7). The closed-loop system resulting from an inexact DMPC is given by

$$\begin{aligned} u(t) &= Kx(t) + v_\epsilon(0|t) = \overline{u}_\epsilon(0|t),\\ x(t+1) &= A_Kx(t) + v_\epsilon(0|t) = \overline{x}_\epsilon(1|t). \end{aligned} \tag{19}$$

Thus, Prop. 6 implies that the closed loop based on an ϵ-feasible solution satisfies the state and input constraints.

Remark 7 *In order to show feasibility of the consolidated trajectory, the constraint tightening* (11),(12) *could be formulated without the term* $k(\epsilon_x + \epsilon_\lambda)$ *and the support function could be defined based on the smaller set* $\mathcal{W}_0 \times \cdots \times \mathcal{W}_k$*, compare [7]. The more restrictive constraint tightening will be crucial in order to establish recursive feasibility of Problem* (14) *for the closed-loop system* (19) *based on an* ϵ*-feasible solution. The issue of using a more conservative constraint tightening to establish recursive feasibility is also addressed in [19,35].*

3.3 Recursive feasibility under inexact minimization

The following Theorem is the main contribution of this paper. It establishes recursive feasibility of Problem (14) under the inexact MPC control law with a suitable candidate solution.

Theorem 8 *Let Assumptions 1 and 4 hold. Given an* ϵ*-feasible solution* (15) $z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)$ *at time* t*, the candidate sequence*

$$\begin{aligned} \tilde{v}(k|t+1) &= v_\epsilon(k+1|t), k = 0,\dots,N-2, \\ \tilde{v}(N-1|t+1) &= (K_f - K)\tilde{z}(N-1|t+1),\\ \tilde{z}(0|t+1) &= x(t+1) = z_\epsilon(1|t) + w_0, \quad w_0 \in \mathcal{W}_0,\\ \tilde{z}(k|t+1) &= z_\epsilon(k+1|t) + A_K^k w_0, k = 0,\dots,N-1,\\ \tilde{z}(N|t+1) &= (A + BK_f)\tilde{z}(N-1|t+1), \end{aligned} \tag{20}$$

is an ϵ_λ*-strictly feasible solution to the optimization Problem* (14) *at time* $t+1$*. Problem* (14) *is recursively feasible for the closed-loop system* (19)*.*

PROOF. The proof is composed of three parts. First, we show strict satisfaction of the relaxed dynamic constraints. Then we show strict satisfaction of the tightened state and input constraints. Finally, we show strict satisfaction of the terminal constraint and thus establish recursive feasibility.
Part I: Show that the candidate sequence $\tilde{z}(\cdot|t+1), \tilde{v}(\cdot|t+1)$ in (20) strictly satisfies the relaxed dynamic constraint (14b): The candidate input $\tilde{v}(\cdot|t+1)$ (20) is constructed by shifting the previous input sequence $v_\epsilon(\cdot|t)$ by one time step and appending the terminal controller K_f. The state sequence $z_\epsilon(\cdot|t)$ is shifted with an additional error term w_0 propagated through the system dynamics to ensure satisfaction of the initial state constraint (14e). Substituting (17) in $\tilde{z}(\cdot|t+1)$ yields

$$\begin{aligned} &\tilde{z}(k|t+1) = z_\epsilon(k+1|t) + A_K^k w_0\\ &= A_K z_\epsilon(k|t) + Bv_\epsilon(k|t) + A_K^k w_0 + w_k\\ &= A_K\tilde{z}(k-1|t+1) + B\tilde{v}(k-1|t+1) + w_k, \end{aligned}$$

for $k = 1,\dots,N-1$, with $\|w_k\|_\infty \leq \epsilon_{z,k} + \epsilon_z = \epsilon_{z,k-1} - \epsilon_\lambda$. Similarly, the last dynamic constraint ($k = N$) is satisfied with equality, which implies that all relaxed dynamic constraints are strictly satisfied with $\epsilon_{z,N-1} = \epsilon_\lambda$.
Part II: Show that the candidate sequence (20) strictly satisfies the state and input constraints (14c): Due to the

definition of the support function [3] and linear superposition we have

$$\sigma_{\mathcal{W}}(H_j^\top, k+1) \geq \sigma_{\mathcal{W}}(H_j^\top, k) + H_j A_K^k w_0, \quad \forall w_0 \in \mathcal{W}_0,$$

which implies

$$\overline{h}_{j,k+1} + H_j A_K^k w_0 \leq \overline{h}_{j,k} - (\epsilon_x + \epsilon_\lambda).$$

The candidate sequence satisfies

$$\begin{aligned} &H_j \tilde{z}(k|t+1) = H_j z_\epsilon(k+1|t) + H_j A_K^k w_0 \\ &\leq \overline{h}_{j,k+1} + H_j A_K^k w_0 + \epsilon_x \leq \overline{h}_{j,k} - \epsilon_\lambda, \quad k = 0, \dots, N-2. \end{aligned}$$

and hence the state constraints are strictly satisfied. For the input constraints the same argument holds with

$$\begin{aligned} &L_j(\tilde{v}(k|t+1) + K\tilde{z}(k|t+1)) \\ &= L_j(v_\epsilon(k+1|t) + K z_\epsilon(k+1|t) + K A_K^k w_0) \\ &\leq \overline{l}_{j,k+1} + \epsilon_u + L_j K A_K^k w_0 \leq \overline{l}_{j,k} - \epsilon_\lambda, \quad k = 0, \dots, N-2. \end{aligned}$$

Given $z_\epsilon(N|t) \in \overline{\mathcal{X}}_{f,\epsilon}$, conditions (13b) and (13c) imply strict satisfaction of the state and input constraints at $k = N-1$.
Part III: Show that the terminal state of the candidate sequence (20) strictly satisfies the terminal constraint (14d): Condition (13d) ensures

$$\begin{aligned} &\tilde{z}(N|t+1) \\ &= (A + BK_f)\tilde{z}(N-1|t+1) \\ &= (A + BK_f)(z_\epsilon(N|t) + A_K^{N-1} w_0) \\ &\in (A + BK_f)(\overline{\mathcal{X}}_{f,\epsilon} \oplus A_K^{N-1}\mathcal{W}_0) \subseteq \overline{\mathcal{X}}_{f,\lambda} \subset \overline{\mathcal{X}}_f. \qquad \square \end{aligned}$$

This theorem ensures recursive feasibility under inexact dual optimization with bounded constraint violation. The candidate solution $\tilde{z}$ with the corresponding tightened (and shifted) constraint set $\overline{\mathcal{X}}_k$ is sketched in Figure 1. The tightened constraint set $\overline{\mathcal{X}}_k$ is constructed, such that ϵ-feasibility of $z_\epsilon(\cdot|t)$ implies ϵ_λ-strict feasibility of $\tilde{z}(\cdot|t+1)$ w.r.t. the shifted constraint set $\overline{\mathcal{X}}_k$, despite the error w_0.

3.4 Closed-loop stability

To study stability properties of the closed-loop system, we use the following definition regarding the suboptimality of the inexact solution.

[3] This would not hold, if we would use $\mathcal{W}_0 \times \cdots \times \mathcal{W}_{k-1}$ for the definition of the k-step support function, compare Remark 7.

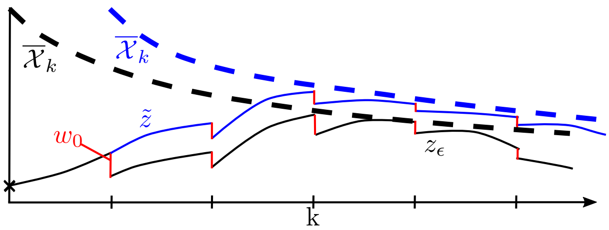

Fig. 1. Illustration of the strictly feasible candidate sequence $\tilde{z}(\cdot|t+1)$ in relation to the previous solution $z_\epsilon(\cdot|t)$, the error in the first dynamic constraint w_0 and the (shifted) tightened constraints $\overline{\mathcal{X}}_k$ over the prediction horizon.

Definition 9 *Given an ϵ-feasible solution (Def. 5), the suboptimality η w.r.t. the optimal solution is defined as*

$$J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq V_N(x(t)) + \eta. \tag{21}$$

The inexact optimal solution is given by

$$\begin{aligned} V_{N,\epsilon}(x(t)) := \min_{z(\cdot|t), v(\cdot|t)} \; & J_N(z(\cdot|t), v(\cdot|t)) \\ \text{s.t. } & z(\cdot|t), v(\cdot|t) \text{ satisfy (15)}. \end{aligned} \tag{22}$$

The suboptimality η_ϵ with respect to this inexact optimal solution is given by

$$J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq V_{N,\epsilon}(x(t)) + \eta_\epsilon. \tag{23}$$

Solutions satisfying (15),(21),(23) *are called* $(\epsilon, \eta, \eta_\epsilon)$-*approximate solutions.*

Corresponding bounds on the suboptimality for inexact dual optimization will be established in Proposition 12. The following proposition shows that the proposed inexact DMPC approximately preserves the stability properties of nominal MPC based on exact optimization.

Proposition 10 *Let Assumptions 1 and 4 hold. Given an $(\epsilon, \eta, \eta_\epsilon)$-approximate solution $z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)$ (Def. 9) at time t, the candidate sequence $\tilde{z}(\cdot|t+1), \tilde{v}(\cdot|t+1)$ in Theorem 8 implies*

$$V_N(x(t+1)) \leq \; V_N(x(t)) - \ell(x(t), v(t)) + \eta. \tag{24}$$

Hence the origin $x = 0$ is practically asymptotically stable [13, Def. 2.15] for the closed-loop system (19) *based on $(\epsilon, \eta, \eta_\epsilon)$-approximate solutions at each time t. Given a sufficiently small ϵ, η_ϵ, the additional bound*

$$V_{N,\epsilon}(x(t+1)) \leq \; V_{N,\epsilon}(x(t)) - \ell(x(t), v(t)) + \eta_\epsilon + \beta_1 \tag{25}$$

holds with β_1 according to (26).

PROOF. Part I: Consolidated cost V_N: The candidate input sequence $\tilde{v}(\cdot|t+1)$ from Theorem 8 with the corresponding consolidated state trajectory $\overline{x}(\cdot|t+1)$ is a feasible solution to (7) (Prop. 6). Using suboptimality η according to Definition 9, this implies

$$\begin{aligned} &V_N(x(t+1)) \leq J_N(\overline{x}(\cdot|t+1), v(\cdot|t+1)) \\ &\overset{(6a)}{\leq} J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) - \ell(x(t), v(t)) \\ &\overset{(21)}{\leq} V_N(x(t)) + \eta - \ell(x(t), v(t)). \end{aligned}$$

Practical asymptotic stability follows from standard Lyapunov arguments.

Part II: Inexact optimal cost $V_{N,\epsilon}$: There exist constants $\underline{\alpha}$, $\overline{\alpha}$, such that $z_\epsilon \in \overline{\mathcal{X}}_f$ implies $V_f(z_\epsilon) \leq \overline{\alpha}$, and $V_f(z_\epsilon) \leq \underline{\alpha}$ implies $z_\epsilon \in \overline{\mathcal{X}}_f$. In the following we consider a bound $V_{N,\epsilon}(x(t)) \leq \overline{V}_\epsilon - \eta_\epsilon$, with some $\overline{V}_\epsilon \geq \underline{\alpha} + \eta_\epsilon$, which is recursively established in the end. This bound in combination with the suboptimality implies $J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}_\epsilon$. The stage cost and terminal cost of the candidate solution satisfy

$$\begin{aligned} &\ell(\tilde{z}(k|t+1), \tilde{v}(k|t+1)) - \ell(z_\epsilon(k+1|t), v_\epsilon(k+1|t)) \\ &\leq \|A_K^k w_0\|_Q^2 + 2\sqrt{\overline{V}_\epsilon}\|A_K^k w_0\|_Q, \quad k = 0, \ldots, N-1, \\ &V_f(\tilde{z}(N-1|t)) - V_f(z_\epsilon(N-1|t)) \\ &\leq \|A_K^{N-1} w_0\|_P^2 + 2\sqrt{\overline{\alpha}}\|A_K^{N-1} w_0\|_P. \end{aligned}$$

The cost of the candidate trajectory satisfies

$$\begin{aligned} &J_N(\tilde{z}(\cdot|t+1), \tilde{v}(\cdot|t+1)) - J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \\ &\overset{(6a)}{\leq} -\ell(x(t), v(t)) + V_f(\tilde{z}(N-1|t+1)) - V_f(z_\epsilon(N|t)) \\ &+ \sum_{k=0}^{N-2} \ell(\tilde{z}(k|t+1), v_\epsilon(k+1|t)) - \ell(z_\epsilon(k+1|t), v_\epsilon(k+1|t)) \\ &\leq -\ell(x(t), v(t)) + \beta_1, \end{aligned}$$

with

$$\begin{aligned} \beta_1 := &\sum_{k=0}^{N-2} \|A_K^k w_0\|_Q^2 + 2\sqrt{\overline{V}_\epsilon}\|A_K^k w_0\|_Q \\ &+ \|A_K^{N-1} w_0\|_P^2 + 2\sqrt{\overline{\alpha}}\|A_K^{N-1} w_0\|_P^2. \end{aligned} \tag{26}$$

Using feasibility based on Theorem 8 and suboptimality η_ϵ according to Definition 9, this implies

$$\begin{aligned} &V_{N,\epsilon}(x(t+1)) \leq J_N(\tilde{z}(\cdot|t+1), \tilde{v}(\cdot|t+1)) \\ &\leq J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) - \ell(x(t), v(t)) + \beta_1 \\ &\overset{(23)}{\leq} V_{N,\epsilon}(x(t)) + \eta_\epsilon + \beta_1 - \ell(x(t), v(t)). \end{aligned}$$

The upper bound $\overline{V}_\epsilon$ is valid recursively, if ϵ, η_ϵ are sufficiently small, such that the following inequality holds $\overline{V}_\epsilon - \eta_\epsilon \geq \underline{\alpha} \geq \lambda_{\max}(P/Q)(\beta_1 + \eta_\epsilon)$, compare [23, Lemma 7, Thm. 8]. □

Theorem 8 in combination with Proposition 10 ensures recursive feasibility and practical asymptotic stability under inexact dual optimization with bounded constraint violation and suboptimality. Both inequalities (24), (25) are each independently sufficient for practical asymptotic stability with the corresponding value functions V_N, $V_{N,\epsilon}$ as practical Lyapunov functions. The stability analysis based on $V_{N,\epsilon}$ tends to be less conservative (compare Prop. 12) and is only possible since we explicitly refrain from adapting the accuracy ϵ online, contrary to [7,9,29]. This is why we also prove the technically more difficult, but potentially less conservative, bounds on the inexact value function (25).

Remark 11 *Due to the inexact dynamic constraint* (14b)*, the input $v_\epsilon = 0$ is the optimal solution to Problem* (14) *if $\|x(t)\|$ is small enough. This property is another reason why the additional feedback in* (4) *can be advantageous.*

3.5 Dual distributed optimization

In the following, we describe how to obtain an approximate solution to Problem (14) with finite dual distributed iterations. Problem (14) can be formulated [4] as

$$\begin{aligned} &\min_y \frac{1}{2}\sum_{i\in\mathcal{N}} \|y_i\|_{\overline{Q}_i}^2 \\ &\text{s.t.} \sum_{j\in\mathcal{N}_i} C_{ij} y_j \leq c_i, \quad i \in \mathcal{N}, \\ &\qquad y_i = (v_i(0|t), z_i(1|t), \ldots v_i(N-1|t), z_i(N|t)),\ c \in \mathbb{R}^{n_c}. \end{aligned}$$

The local dual gradient is Lipschitz with $L_{d_i} = \frac{\|[C_{ji}]_{j\in\mathcal{N}_i}\|^2}{\lambda_{\min}(\overline{Q}_i)}$. We consider the distributed dual gradient algorithm [30], with the local step size $W_{\mu,i} = \sum_{j\in\mathcal{N}_i} L_{d_j}$.

Algorithm 1. Distributed Dual Gradient (**DDG**)
Given Lipschitz constant $W_{\mu,i}$ (computed offline)
Initialization: set initial guess μ^0
1: **for** $p = 0, \ldots \overline{p}$ **do**
2: $y_i^{p+1} = -\overline{Q}_i^{-1} \sum_{j\in\mathcal{N}_i} C_{ji}^\top \mu_j^p$
3: $\mu_i^{p+1} = [\mu_i^p + W_{\mu,i}^{-1}(\sum_{j\in\mathcal{N}_i} C_{ij} y_j^{p+1} - c_i)]_+$
4: **end for**

Here $[\,]_+$ denotes the projection on the nonnegative orthant. This is an iterative synchronous algorithm, which consists of small-scale multiplications and requires only local communication. The following proposition summarizes the theoretical properties of this algorithm.

Proposition 12 *Suppose there exists an ϵ_λ-strictly feasible solution $\tilde{y}$ to Problem* (14) *and consider the ini-*

[4] The minimization can be further decoupled along the time axis with the variables $y_{i,k} = [v_i(k|t), z_i(k|t)]$, compare [7].

tialization[5] $\mu^0 = 0$. *For all* $p \geq \overline{p}$, y^p *is an* $(\epsilon, \eta, \eta_\epsilon)$-*approximate solution (Def. 5, 9) with*

$$\overline{p} = 2 + \log_d[\epsilon_\lambda \, \underline{\epsilon} \, \|W_\mu^{-1}\|/\overline{V}_\epsilon], \quad d < 1, \tag{27a}$$

$$\eta_\epsilon = \sqrt{n_c}\, \overline{V}_\epsilon \, \overline{\epsilon}/\epsilon_\lambda + \frac{1}{2}\underline{\epsilon}^2 \|W_\mu^{-1}\|, \tag{27b}$$

$$\eta = \sqrt{n_c}\, \overline{V}_\epsilon \, \overline{\epsilon}_N/\epsilon_\lambda + \frac{1}{2}\underline{\epsilon}^2 \|W_\mu^{-1}\|, \tag{27c}$$

$\overline{\epsilon}_N = \max\{\overline{\epsilon}, \epsilon_{z,0}, \overline{h}_{j,N} - h_j, \overline{l}_{j,N} - l_j\}$, $\underline{\epsilon} = \min\{\epsilon_z, \epsilon_x, \epsilon_u, \epsilon_f\}$, $\overline{\epsilon} = \max\{\epsilon_z, \epsilon_x, \epsilon_u, \epsilon_f\}$, $\overline{V}_\epsilon = \|\tilde{y}\|^2_{\overline{Q}}$.

PROOF. This result is based on [30, Thm. 4.2-4.4] and strict feasibility.
Part I: Given the ϵ_λ-strictly feasible solution $\tilde{y}$, the following upper bound holds for the optimal dual variable μ^* (compare [7, Lemma 1])

$$\|\mu^*\| \leq \frac{\|\tilde{y}\|^2_{\overline{Q}} - \|y^*\|^2_{\overline{Q}}}{\epsilon_\lambda} \leq \frac{\|\tilde{y}\|_{\overline{Q}^2}}{\epsilon_\lambda} \leq \overline{V}_\epsilon/\epsilon_\lambda.$$

Based on [30, Thm. 4.2] the constraint violation satisfies

$$\|[Cy^p - c]_+\| \leq d^{p-2}\, \overline{V}_\epsilon/\epsilon_\lambda \|W_\mu\|,$$

with the fixed constant $d < 1$ according to [30, Thm. 3.2, Thm. 4.2]. Correspondingly, for $p \geq \overline{p}$, y^p is an $\epsilon-$feasible solution, with $\overline{p}$ according to (27a).
Part II: Analogous to [7, Thm. 1], we can derive the following bound based on the dual variables and the relaxation

$$V_{N,\epsilon}(x(t)) \leq J_N(z^*(\cdot|t), v^*(\cdot|t)) + \sqrt{n_c}\, \overline{V}_\epsilon \, \overline{\epsilon}/\epsilon_\lambda,$$

where z^*, v^* is the optimal solution to (14). By combining this result with the relative suboptimality η_y of y^p ([30, Thm. 4.4]), we can establish the following bound

$$\eta_\epsilon \leq \|\mu^*\|_1 \, \overline{\epsilon} + \eta_y \leq \sqrt{n_c}\, \overline{V}_\epsilon \, \overline{\epsilon}/\epsilon_\lambda + \eta_y,$$
$$\eta_y := \frac{1}{2}(d^{\overline{p}-1}\, \overline{V}_\epsilon/\epsilon_\lambda)^2 \|W_\mu\| \leq \frac{1}{2}\underline{\epsilon}^2 \|W_\mu^{-1}\|.$$

The same derivations hold for η, using the maximal size of the constraint tightening $\overline{\epsilon}_N$ instead of $\overline{\epsilon}$. □

Remark 13 *Given a user specified accuracy* ϵ, *this proposition gives an a priori upper bound on the number of iterations* $\overline{p}$ *for a given sublevel set of* $V_{N,\epsilon}$. *In combination with Theorem 8 and Proposition 10, this property holds recursively under the approximate DMPC. In closed-loop operation the value of* $\overline{V}_{N,\epsilon}$ *decreases (Prop. 10) and thus the number of necessary iterations* $\overline{p}$ *based on* (27a) *decreases. Using a larger tolerance* ϵ *leads to fewer iterations p and a larger suboptimality* η, η_ϵ. *The bound for* η_ϵ *is (typically) significantly smaller than* η, *which is crucial for the stability analysis (Prop. 10). Instead of choosing a desired accuracy* ϵ, *a user can also specify an upper bound on the number of iterations* $\overline{p}$ *and choose a sufficiently small accuracy* ϵ *using* (27a). *There exists a variety of distributed dual algorithms for which similar complexity bounds can be obtained. If an alternating minimization algorithm such as [17,7] is used, the relationship between the inexactness of the optimization* ϵ *and the resulting inexactness in the dynamic constraint changes, see [7,24].*

[5] The following properties remain valid if the initialization μ^0 satisfies $\|\mu^0 - \mu^*\|_W \leq \|\mu^*\|_W$.

The initialization and closed-loop operation of the MPC scheme is summarized in the following two algorithms.

Algorithm 2. Offline distributed synthesis
1. Set design parameters $\epsilon_z,\ \epsilon_x,\ \epsilon_u,\ \epsilon_f,\ \epsilon_\lambda \in \mathbb{R}_{>0}$.
2. Compute tightened constraints (11), (12).
3. Compute terminal cost and set (Ass. 4).
4. Compute Lipschitz constant W_{μ_i} (for Alg. 1).

Algorithm 3. Online DMPC, execute at every time step t
1. Measure the state $x(t)$.
2. Compute candidate dual variable μ^0.
3. Approximately solve Problem (14) with Alg. 1.
4. Apply control input: $u_i(t) = K_{\mathcal{N}_i} x_{\mathcal{N}_i}(t) + v_{i,\epsilon}(0|t)$.

Instead of using the (possibly conservative) a priori bound $\overline{p}$, a stopping condition ensuring an ϵ-feasible solution (Def. 5) can be used, which can be efficiently and distributedly checked online, compare Section 4. All the necessary offline and online computations can be accomplished in a fully distributed and scalable fashion.

3.6 Comments

By combining Theorem 8 and Propositions 10 and 12, we can ensure recursive feasibility and practical asymptotic stability with finite distributed dual iterations. While parts of the proof might be technical, the application of the proposed method is straightforward. The bounds on the suboptimality η_ϵ, η and the resulting closed-loop stability guarantees (Prop. 10, Prop. 12) tend to be conservative and should rather be interpreted as a conceptual result of how the inexact minimization affects stability.

We prove the theoretical properties of the proposed framework within the standard MPC setup including a terminal cost and a polytopic terminal set. In various applications and setups, different variations of MPC can be advantageous (such as MPC without terminal ingredients or economic MPC). The Appendix shows in detail under which conditions similar results can be derived for these different setups.

The following remark discusses similarities of the proposed framework to existing schemes and highlights the novelty based on the inexact candidate solution.

Remark 14 *In [35] violations in the inequality constraints (state and input constraints ϵ_x, ϵ_u) are considered. The corresponding constraint tightening can be viewed as a special case ($\epsilon_z = 0$) of the proposed method.*

In [19] a constraint tightening is proposed to ensure recursive feasibility of the consolidated trajectory despite inexact dynamic constraints. The a priori complexity bounds (Prop. 12) do not hold for this formulation due to the usage of equality constraints and lack of strict feasibility.

In [9] the stopping condition is based on an explicit candidate solution for the next time step, which needs to be computed online. This requires additional online computations and bounds on the number of iterations cannot be given (in contrast to ϵ-feasibility as used in [35,29,7]).

In [29,7] the constraints are tightened, such that the consolidated trajectory is (strictly) feasible (Prop. 6). Recursive feasibility is ensured be adapting the accuracy ϵ and constraint tightening online. This adaptation requires global communication, is complex, and it is a priori unclear whether the number of online iterations increases or decreases in closed-loop operation. One of the main benefits of the proposed framework is that such an adaptation is not needed (although incorporating an optional adaptation, if possible, could be beneficial).

To the best of our knowledge, the proposed result is the first MPC result based on a dynamic inexact candidate solution. As discussed above, the use of such an inexact candidate solution is possible through relaxing the dynamic constraint (14b) *and is the key ingredient for establishing recursive feasibility with a fixed constraint tightening, allowing for a fully distributed implementation of the proposed scheme with finite dual iterations.*

4 Numerical Example

In the following, we show the practicality of the proposed approach with the example [6] of a chain of masses [3]. We consider $M = 20$ subsystems with randomly sampled mass $m \in [0.5, 1.5]$, spring constant $k \in [1.5, 4.5]$, damping constant $d \in [1.5, 4.5]$ and use an Euler discretization with $h = 0.1$ s. The cost is $Q = I$, $R = I$ and the constraints are $\|u_i\|_\infty \leq 10$, $|[1,0]x_i| \leq 10$, $|[0,1]x_i| \leq 1$, $\|x_i - x_j\|_\infty \leq 3$, $j \in \mathcal{N}_i$. The resulting system has 40 states, 20 inputs, and coupled dynamics and constraints.

[6] To improve the numerical conditioning, we set $\tilde{u} = 10 \cdot u$.

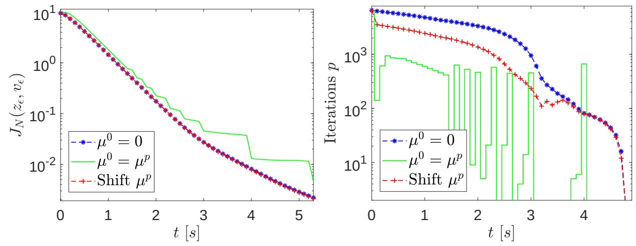

Fig. 2. Closed-loop Inexact DMPC: Inexact cost $J_N(z_\epsilon, v_\epsilon)$ (left) and number of iterations p (right) with different initialization μ^0 vs. time t.

Offline computation

In the following we detail the offline computations. We consider no additional feedback, i.e., $K = 0$. We choose a prediction horizon of $N = 4$ and the tolerance is chosen as $\epsilon = 2.5 \cdot 10^{-3} = \epsilon_z = \epsilon_x = \epsilon_u = \epsilon_\lambda = \epsilon_f$. The constraints are tightened with the k-step support function (11), (12). We compute a distributed terminal cost P that satisfies (6a) with distributed LMIs as in [3]. For the terminal set, we consider decoupled local terminal sets $\overline{\mathcal{X}}_f = \overline{\mathcal{X}}_{f,1} \times \ldots \overline{\mathcal{X}}_{f,M}$, with symmetric local sets

$$\overline{\mathcal{X}}_{f,i} = \{x_i | \; Gx_i \leq q_i, \; -Gx_i \leq q_i\}, \; G^\top = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}.$$

The vectors q_i are determined using the method in [38], such that the set $\overline{\mathcal{X}}_{f,\epsilon}$ is (robust) positively invariant for the dynamics $A + BK_f$, by solving a (distributed) LP. This terminal set is scaled, such that the conditions (13c),(13b) are satisfied. Finally, we verify that this terminal set satisfies condition (13d) and thus Assumptions 1, 4 are satisfied. The overall offline computations are accomplished in 60 s with an Intel Core i7.

Simulations - Stability and dual initialization

In the following, the online optimization (Alg. 1) is stopped once an ϵ-feasible solution (Def. 5) is obtained. We explore the effect of the initialization μ^0 (Alg. 3) on the number of dual iterations p. Simple initialization strategies are $\mu^0 = 0$, using the previous dual variables $\mu^0 = \mu^p$, or shifting μ^p similar to the shifted candidate solution $\tilde{z}$ in Theorem 8 (and appending zero at the end). We consider an initial condition with random positions and zero velocity. The inexact cost $J_N(z_\epsilon, v_\epsilon)$ and number of dual iterations p for the resulting closed loop can be seen in Figure 2. As expected, the predicted cost J_N decreases and the origin is (practically) asymptotically stable (Prop. 10). Clearly, using the previous solution μ^p can significantly reduce the number of online iterations. Since the cost J_N with different initialization μ^0 only varies marginally, a suitable initialization simply reduces the number of online iterations.

In the following, we quantitatively explore the effect of the tolerance ϵ and the number of subsystems M on the

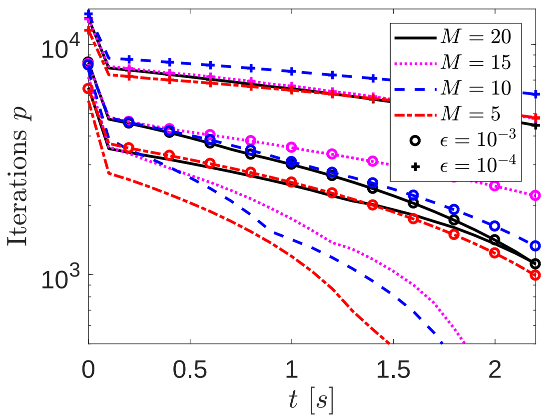

Fig. 3. Quantitative impact of tolerance ϵ and number of subsystems M on number of iterations p.

closed-loop computational demand. We consider the initialization based on the shifted vector μ^p. In Figure 3 we can see the number of online iterations p in each time step for different numbers of subsystems $M \in \{5, 10, 15, 20\}$ and tolerances $\epsilon \in \{2.5 \cdot 10^{-3}, 10^{-3}, 10^{-4}\}$. If we increase the number of subsystems M, the number of dual iterations tends to increase slightly (due to the increased cost $\overline{V}_\epsilon$, compare Prop. 12). In contrast, if we chose a smaller tolerance ϵ the number of dual iterations increases significantly. Thus, by choosing a larger tolerance ϵ, we can consider significantly more subsystems M without increasing the number of online dual iterations p.

To summarize: Compared to a nominal DMPC, the design procedure only requires the additional computation of the tightened constraints for the chosen tolerance ϵ. With the proposed modifications, the closed loop satisfies the constraints and the effect of inexact minimization on closed-loop stability is negligible. Thus, we can significantly reduce the online computational demand by allowing for a non-vanishing tolerance ϵ without any major downside. The main limitation to consider more subsystems M and a larger tolerance ϵ is the construction of the polytopic terminal set $\overline{\mathcal{X}}_f$ and the robust positive invariance condition (13d).

5 Conclusion

We have proposed a new formulation for DMPC based on inexact dual optimization. The online optimization can be accomplished in a fully distributed manner using standard dual distributed optimization methods and only has to obtain an approximate solution. We have established recursive feasibility, constraint satisfaction and practical stability of the closed loop based on such an approximate solution. This is possible through the usage of a reformulated optimization problem and a novel candidate solution, which both explicitly consider the inexactness of the optimization. This modified formulation enables practical applications of MPC to large-scale systems with fast dynamics, for which the underlying MPC optimization problem cannot be solved in real time.

Acknowledgements

Preliminary results of this paper have been derived during a stay of the first author in Na Li's research group in SEAS Harvard. The authors thank the German Research Foundation (DFG) for support of this work within grant AL 316/11-1 and within the Research Training Group Soft Tissue Robotics (GRK 2198/1).

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A Extensions

This appendix discusses the distributed offline computation and details how the proposed framework (in particular Thm. 8 and Prop. 10) can be extended to different MPC setups. In Section A.1 the distributed offline computation is discussed and a novel algorithm for the computation of the distributed polytopic terminal set is presented. In Section A.2 MPC without explicit terminal constraints is discussed. Next, in Section A.3 we show that the closed-loop properties can be guaranteed without using a terminal cost. In Section A.4 these results are extended to unreachable setpoints and economic MPC. Section A.5 shows how using multi-step MPC can be beneficial in the proposed framework. Finally, Section A.6 discusses the necessary steps to extend the framework to nonlinear systems.

Notation

We require the following additional notation. For positive definite matrices $P,\ Q$ we denote the minimal and maximal eigenvalue for the generalized eigenvalue problem $(P-\lambda_j Q)v_j=0$ by $\lambda_{\min}(P/Q)$ and $\lambda_{\max}(P/Q)$, respectively. By $\mathcal{K}$ we denote the class of functions $\alpha:\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$, which are continuous, strictly increasing and satisfy $\alpha(0)=0$. We denote the class of functions $\delta:\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$, which are continuous and decreasing with $\lim_{\rightarrow\infty}\delta(k)=0$ by $\mathcal{L}$.

A.1 Distributed offline computation

In order to implement the proposed distributed scheme, a block-diagonal terminal cost P in combination with a distributed feedback K_f is needed (Ass. 1). Furthermore, a distributed polytopic terminal set $\overline{\mathcal{X}}_f$ (Ass. 4) is necessary. In the following, we briefly summarize how these offline computation can be accomplished is a scalable distributed fashion.

Distributed terminal cost

In a centralized setting, Assumption 1 is standard when using an MPC framework including a terminal cost and terminal set [34]. Concerning the distributed setting of this paper, in [3] distributed linear matrix inequalities (LMIs) are presented that can be used to compute a distributed terminal cost P. Once a stabilizing controller K_f is given, it is also possible to compute a less conservative terminal cost of the form $V_f(x)=\sum_{i\in\mathcal{N}}x_{\mathcal{N}_i}P_i x_{\mathcal{N}_i}$, compare [4]. We need to consider some distributed structure for $P,\ K_f$ to allow for a scalable offline and online optimization. Correspondingly, the procedures are only sufficient conditions. If it is not possible to compute such a distributed matrix P, the terminal cost can also be omitted, which is discussed in Section A.3.

Distributed polytopic terminal sets

In order to satisfy Assumption 4, we need to compute a distributed (robust) positively invariant polytope $\overline{\mathcal{X}}_f$. In a central setting, computing a (robust) positively invariant polytope can be done using standard methods, compare [8,32]. Computing a distributed positively invariant polytopic set is largely still an open problem. In the following we discuss one method based on [38] to compute a distributed polytopic terminal set. First, we choose a set of r half spaces F_j, that parameterize a terminal set of the form $\overline{\mathcal{X}}_f=\{x|\ Fx\leq\overline{f}\}$. By choosing F_j, we can impose any desired distributed structure on $\overline{\mathcal{X}}_f$. Given F, the method in [38] presents a linear program (LP) to compute $\overline{f}$, such that $\overline{\mathcal{X}}_f$ is robust positively invariant for the dynamics $A+BK_f$ and a disturbance set $\mathcal{W}$. If $F,\ A+BK_f$ and $\mathcal{W}$ have a distributed structure, the resulting LP also has a distributed structure and can be solved efficiently. Thus, we can compute a distributed polytopic terminal set, satisfying Assumption 4. The main limitation of this method is the fact that the half-spaces F_j need to be chosen in advance by the user. There is in general no guarantee that a distributed polytopic terminal set exists and how complex (number of vertices/half-spaces) the local polytopes need to be. Thus, we also briefly discuss some alternatives.

Given the block-diagonal terminal cost P, it is possible to use (distributed) ellipsoidal terminal sets $\overline{\mathcal{X}}_f$ which satisfy Assumption 1, compare [3]. However, the drawback is that the ellipsoidal terminal constraints lead to a (distributed) quadratically constrained quadratic program (QCQP), which makes the online optimization more complex.

The requirement of positive invariance can also be relaxed. In particular, given a Schur stable matrix $\tilde{A}$, any compact (distributed) polytopic set $\overline{\mathcal{X}}_f$ is positively invariant for the system $\tilde{A}^M$, with M large enough. In [17] this is used to construct a distributed polytopic terminal set for multi-step MPC, which is discussed in Section A.5.

If we are unable to construct a suitable distributed terminal constraint, the terminal constraint can also be implicitly enforced, which is discussed in Section A.2.

Distributed robustly stabilizing feedback

The proposed framework is such that the MPC input v can be augmented by an additional (optional) feedback K, compare (4). If we do not know any suitable feedback, we can always choose $K=0$. If Assumption 1 is satisfied, one natural choice for this feedback is the terminal controller, i.e., $K=K_f$, which also simplifies some formulas. It is also possible to compute a distributed feedback K, which is specifically designed to reduce the constraint tightening. In particular, by extending the procedure in [25] one can compute a distributed feedback K that minimizes the resulting constraint tightening for a given disturbance w, see [20, Sec. 2.4] for details. We would like to point out, that although the feedback K can reduce conservatism in the constraint tightening, it may also have an adversarial impact on the conditioning of the optimization problem.

A.2 MPC without explicit terminal constraints

Computing a polytopic distributed terminal set (Ass. 1,4) can be difficult (App. A.1), if not impossible. Given a block diagonal terminal cost P, it is possible to obtain a suitable distributed ellipsoidal constraint, compare [3]. However, including such quadratic constraints increases the online computational complexity. The following proposition shows that $(\epsilon, \eta, \eta_\epsilon)$-approximate solutions implicitly satisfy the terminal constraints if the terminal cost P or the prediction horizon N are chosen large enough, similar to [26].

Proposition 15 *Let Assumptions 1 and 4 hold. There exist constants α, $\overline{\alpha} \in \mathbb{R}_{>0}$, such that $\|x\|_P^2 \leq \alpha$ and $\|z\|_P^2 \leq \overline{\alpha}$ implies $x \in \mathcal{X}_f$ and $z \in \overline{\mathcal{X}}_f$, respectively. Consider Problem* (14) *without an explicit terminal constraint. If the open-loop cost satisfies*

$$J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V} := \alpha(1 + \lambda_{\min}(Q/P)N),\\ \eta \leq \lambda_{\min}(Q/P)\alpha,$$

then $\overline{x}_\epsilon(N|t) \in \mathcal{X}_f$. Similarly,

$$J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}_\epsilon := \overline{\alpha}(1 + \lambda_{\min}(Q/P)N),\\ \eta_\epsilon \leq \lambda_{\min}(Q/P)\overline{\alpha},$$

implies $z_\epsilon(N|t) \in \overline{\mathcal{X}}_{f,\epsilon}$.

PROOF. We proof this property for the consolidated trajectory $\overline{x}_\epsilon$ with the nominal terminal set $\mathcal{X}_f$. The proof regarding the inexact trajectory z_ϵ is analogous.

The implicit satisfaction of the terminal constraint at some point k_x is a standard result [26, Lemma 2], which is based on the bounded cost J_N and the property that $\|\overline{x}_\epsilon(k_x|t)\|_Q^2 \leq \alpha\lambda_{\min}(Q/P)$ implies $\overline{x}_\epsilon(k_x|t) \in \mathcal{X}_f$. Due to suboptimality η, the terminal constraint is not obviously satisfied at the terminal state (contrary to [26, Lemma 1]). With the suboptimality η, the positive definite stage cost and the properties of the terminal cost V_f (Ass. 1), we have

$$\sum_{k=0}^{k_x} \ell(\overline{x}_\epsilon(k|t), v_\epsilon(k|t)) + V_f(\overline{x}_\epsilon(N|t))\\ \leq J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq V_N(x(t)) + \eta\\ \leq \sum_{k=0}^{k_x-1} \ell(\overline{x}_\epsilon(k|t), v_\epsilon(k|t)) + V_f(\overline{x}_\epsilon(k_x|t)) + \eta.$$

By canceling the first k_x components, this is equivalent to

$$V_f(\overline{x}_\epsilon(N|t)) \leq \eta + V_f(\overline{x}_\epsilon(k_x|t)) - \ell(\overline{x}_\epsilon(k_x|t), v_\epsilon(k_x|t))\\ \leq \eta + (1 - \lambda_{\min}(Q/P))V_f(\overline{x}_\epsilon(k_x|t))\\ \leq \eta + \alpha(1 - \lambda_{\min}(Q/P)) \leq \alpha,$$

where the last inequality follows if we assume $\eta \leq \lambda_{\min}(Q/P)\alpha$. This in turn implies $\overline{x}_\epsilon(N|t) \in \mathcal{X}_f$. □

Proposition 15 shows that the consolidated trajectory and the inexact trajectory satisfy the terminal constraint without an explicit constraint [7], if the terminal cost P or the prediction horizon N is chosen large enough. The guarantees in Section 3 remain valid, if the accuracy ϵ is such that (24),(25) ensure positive invariance of the sublevel sets $\overline{V}$, $\overline{V}_\epsilon$. Increasing the terminal cost may, however, increases the number of online iterations and might deteriorate the stability results of the inexact DMPC.

A.3 MPC without terminal cost

Depending on the strength of the coupling in the distributed system dynamics, there might not exist a distributed terminal cost, which satisfies Assumption 1. In addition, offline computations might be unsuitable for changing operation points or plug-and-play setups. The following Proposition shows that stability and recursive feasibility can be ensured under inexact minimization even if there does not exist a structured terminal cost, which is an extension of the nominal results in [10], compare also [9].

Proposition 16 *Assume that (A, B) is stabilizable. Consider Problem* (14) *without terminal constraint and without terminal cost ($V_f = 0$). For any $\overline{V}_\epsilon$, $\overline{V} \in \mathbb{R}_{>0}$ there exists an $N_0 \in \mathbb{N}$, such that for all $N \geq N_0$ and ϵ sufficiently small, there exist α_N, $\beta_2 \in \mathbb{R}_{>0}$, such that for any $(\epsilon, \eta, \eta_\epsilon)$-approximate solution at time t satisfying*

$$J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}_\epsilon, \quad J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}, \tag{A.1}$$

[7] Depending on the algorithm, ensuring a tight bound on η, η_ϵ might be difficult. A practical approach is to include the condition $\|z(N|t)\|_P^2 \leq \overline{\alpha}$ in the sufficient stopping condition.

the closed-loop system (19) *satisfies*

$$V_{N,\epsilon}(x(t+1)) \leq V_{N,\epsilon}(x(t)) - \alpha_N \ell(x(t),v(t)) + \beta_2 + \eta_\epsilon, \quad \text{(A.2)}$$

$$V_N(x(t+1)) \leq V_N(x(t)) - \alpha_N \ell(x(t),v(t)) + \eta. \quad \text{(A.3)}$$

For sufficiently small β_2, η_ϵ, η *and large enough* α_N, *the bounds* (A.1) *remain valid recursively for all* $t \geq 0$ *and the origin* $x = 0$ *is practically asymptotically stable for the closed-loop system* (19) *based on an* $(\epsilon, \eta_\epsilon)$*-approximate solution.*

PROOF. The analysis is analogous to the nominal guarantees as detailed in [22, Thm. 10] based on [8] [10, Variant 1].

Since (A, B) is stabilizable and the origin is strictly inside the constraints, there exists a constant $\alpha > 0$, such that the terminal set $\mathcal{X}_f = \{\|x\|_P^2 \leq \alpha\}$ and the matrcies P, K_f based on the discrete-time linear quadratic regulator (DLQR) satisfy the conditions (6) in Assumptions 1. Note, that we do not need to know P, K_f and in general these matrices are not distributed or block diagonal. Similarly, for a given prediction horizon N, there exists a small enough accuracy ϵ, such that the terminal set $\overline{\mathcal{X}}_f = \{\|x\|_P^2 \leq \overline{\alpha}\}$ with $\overline{\alpha} > 0$ satisfies the conditions (13) in Assumption 4.

Let us define $\gamma_\epsilon := \frac{\overline{V}_\epsilon}{\overline{\alpha}} \lambda_{\max}(P/Q)$. Without loss of generality, assume $\overline{V}_\epsilon \geq \overline{\alpha} + \eta_\epsilon$. For $N > \gamma_\epsilon$, there exists a $k_x \geq 1$, with

$$\|z_\epsilon(k_x|t)\|_Q^2 \leq \frac{\overline{V}_\epsilon}{N} \leq \frac{\gamma_\epsilon}{N} \ell(x(t), v(t)),$$

and thus $z_\epsilon(k_x|t) \in \overline{\mathcal{X}}_{f,\epsilon}$. Consider the candidate solution from Theorem 8 with K_f appended for $k \geq k_x$, i.e.

$$\tilde{z}(k+1|t+1) = (A + BK_f)\tilde{z}(k|t+1), \quad k = k_x, \ldots, N-1,$$
$$\tilde{v}(k|t+1) = (K_f - K)\tilde{z}(k|t+1), \quad k = k_x, \ldots, N-1.$$

Using arguments similar to [22, Thm. 10], one can show that the corresponding cost satisfies

$$\begin{aligned} &J_N(\tilde{z}(\cdot|t+1), \tilde{v}(\cdot|t+1)) - J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \\ \leq\ &\underbrace{(1 - \frac{\gamma_\epsilon}{N}(\lambda_{\max}(P/Q) - 1))}_{=:\alpha_N} \ell(x(t), v(t)) + \beta_2, \end{aligned}$$

with $\beta_2 \leq \beta_1$. We can ensure $\alpha_N > 0$, for

$$N \geq N_0 := 1 + \lfloor \gamma_\epsilon(\lambda_{\max}(P/Q) - 1) \rfloor,$$

with the floor function $\lfloor\ \rfloor$. Analogous to Prop. 10, the closed-loop system satisfies (A.2). Recursive satisfaction of $V_{N,\epsilon}(x(t)) \leq \overline{V}_\epsilon - \eta_\epsilon$ is ensured with [23, Lemma 7, Thm. 8] for $\overline{V}_\epsilon - \eta_\epsilon \geq \overline{\alpha} \geq \frac{1}{\alpha_N} \lambda_{\max}(P/Q)(\beta_2 + \eta_\epsilon)$. The proof regarding the consolidated trajectory is analogous, compare the proof of Prop. 10. □

This proposition significantly relaxes the distributed stabilizability condition in Assumption 1. A drawback of this approach is that a potentially significantly larger prediction horizon N is required. We would like to point out, that the sufficiently small value of ϵ is not uniform, i.e., if we consider a larger prediction horizon N we may need to choose a smaller accuracy ϵ. More general, the closed-loop performance with finite dual iterations does not necessarily improve by increasing the prediction horizon N, due to the larger suboptimality η, η_ϵ (compare [20, Sec. 4.4.3]).

A.4 Unreachable setpoints and economic MPC

Another relevant setup includes tracking of a setpoint (x_r, v_r), which is unreachable (due to constraints or dynamics) [33]. This is a special case of economic MPC [28]. Without loss of generality, the optimal reachable setpoint is given by $(x, v) = 0$. We consider the strictly convex (economic) quadratic stage cost

$$\ell_e(x, v) = \|x\|_Q^2 + x^\top q + \|v\|_R^2 + v^\top r, \quad \text{(A.4)}$$

with Q, R positive definite. For this setup, a strong duality condition is satisfied (see [5, Prop. 4.3]) with the positive definite rotated stage cost

$$\begin{aligned} L(x, v) &= \lambda^\top((I - (A + BK_f))x - Bv) + \ell_e(x, v) \\ &= \|x\|_Q^2 + \|u\|_R^2. \end{aligned}$$

Suppose that Assumption 4 is satisfied with the quadratic stage cost $\ell(x, v) = \|x\|_Q^2 + \|v\|_R^2$ and some distributed terminal cost $V_f(x) = \|x\|_P^2$ and distributed terminal controller K_f. Then, based on a distributed linear equality constraint one can compute a unique vector $p_e \in \mathbb{R}^n$ that satisfies

$$p_e^\top(A + BK_f - I) + q + r(K_f - K) = 0.$$

Correspondingly, the economic terminal cost

$$V_e(x) = \|x\|_P^2 + x^\top p_e \quad \text{(A.5)}$$

satisfies

$$V_e((A + BK_f)x) - V_e(x) \leq -\ell_e(x, (K_f - K)x),$$

[8] The proof technique [10, Variant 2/3] cannot be directly used for suboptimal trajectories as studied in this paper.

compare [1]. Thus, the computation of the distributed economic terminal cost can be done analogous to the tracking case in [3], see [20, Sec. 2.2]. The rotated open loop cost $\tilde{J}_N$, the optimal rotated value function $\tilde{V}_N$ and the inexact rotated value function $\tilde{V}_{N,\epsilon}$ are defined analogous to (7), (22) with the rotated stage cost (A.4). Compared to (14), the resulting distributed economic MPC problem considers the economic stage cost ℓ_e and economic terminal cost V_e instead of the positive definite cost ℓ, V_f. The following proposition shows that in this setup, properties similar to Proposition 10 can be ensured.

Proposition 17 *Let Assumptions 1 and 4 be satisfied. Suppose that the economic MPC with the economic terminal cost* (A.5) *and a strictly convex (economic) quadratic stage cost* (A.4) *is used. Given an $(\epsilon, \eta, \eta_\epsilon)$-approximate solution $z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)$ (Def. 9) at time t, the candidate sequence in Proposition 10 implies*

$$\tilde{V}_N(x(t+1)) \leq \tilde{V}_N(x(t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \eta, \tag{A.6}$$

and the origin $x = 0$ is practically asymptotically stable for the closed-loop system (19) *based on an $(\epsilon, \eta, \eta_\epsilon)$-approximate solution. Furthermore, given a sufficiently small ϵ, η_ϵ, the following additional bound holds*

$$\tilde{V}_{N,\epsilon}(x(t+1)) \leq \tilde{V}_{N,\epsilon}(x(t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \tilde{\eta}_\epsilon + \beta_1 \tag{A.7}$$

with β_1, $\tilde{\eta}_\epsilon$ according to (26)*,* (A.8)*.*

PROOF. The candidate trajectory v (Prop. 10) in combination with the economic terminal cost implies

$$V_N(x(t+1)) \leq V_N(x(t)) + \eta - \ell(x(t), v(t)),$$

which is equivalent to (A.6) (compare [1]). The inexact trajectory satisfies the following rotated suboptimality bound

$$\tilde{J}_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \tilde{V}_{N,\epsilon}(x(t)) + \tilde{\eta}_\epsilon,$$
$$\tilde{\eta}_\epsilon = \eta_\epsilon + 2\|\lambda\|_1 \sum_{k=0}^{N-1} (\epsilon_{z,k} + \epsilon_z). \tag{A.8}$$

The rest of the proof is analogous to Prop. 10, resulting in (A.7). □

The following proposition shows that similar stability results can be obtained in this economic setup without terminal constraints or terminal cost.

Proposition 18 *Assume that (A, B) is stabilizable and the set $\mathcal{X}$ is compact. Consider Problem* (14) *with a strictly convex (economic) quadratic stage cost* (A.4) *and without terminal constraints and without terminal cost ($V_e = V_f = 0$). For any $\overline{V}_\epsilon$, $\overline{V} \in \mathbb{R}_{>0}$ there exists an $N_0 \in \mathbb{N}$, such that for all $N \geq N_0$ and ϵ sufficiently small, there exists functions $\theta, \theta_\epsilon \in \mathcal{L}$, such that for any $(\epsilon, \eta, \eta_\epsilon)$-approximate solution at time t satisfying*

$$J_N(z_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}_\epsilon, \quad J_N(\overline{x}_\epsilon(\cdot|t), v_\epsilon(\cdot|t)) \leq \overline{V}, \tag{A.9}$$

the closed-loop system (19) *satisfies*

$$\tilde{V}_N(x(t+1)) \leq \tilde{V}_N(x(t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \theta(N-2) + \eta, \tag{A.10}$$

$$\tilde{V}_{N,\epsilon}(x(t+1)) \leq \tilde{V}_{N,\epsilon}(x(t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \theta_\epsilon(N-2) + \beta_2 + \eta_\epsilon. \tag{A.11}$$

For sufficiently small $\eta, \eta_\epsilon, \beta_2$ and large enough N, the bounds (A.9) *remain valid recursively and the origin $x = 0$ is practically asymptotically stable for the closed-loop system* (19) *based on an $(\epsilon, \eta, \eta_\epsilon)$-approximate solution.*

PROOF. The proof is analogous to the nominal guarantees as detailed in [21, Thm. 4,5] (see also [11]).

Similar to Proposition 16, (A, B) stabilizable implies that there exists a terminal cost P and terminal controller K_f based on the DLQR. Furthermore, for a sufficiently small accuracy ϵ, the terminal set $\mathcal{X}_f = \{\|x\|_P^2 \leq \alpha\}$ and the terminal set $\overline{\mathcal{X}}_f = \{\|x\|_P^2 \leq \overline{\alpha}\}$ with $\overline{\alpha} > 0$ satisfy Assumption 1 and 4, respectively. Note, that linear dynamics, quadratic cost and polytopic constraints imply that the value function is (locally) continuous, i.e., there exists a functions $\alpha_V \in \mathcal{K}$, such that $|V_N(x) - V_N(y)| \leq \alpha_V(\|x\| + \|y\|)$, compare [21, Ass. 4].

Without loss of generality, assume $\overline{V} \geq \alpha + \eta$. Denote the optimal solution to (7) with the rotated stage cost by $(\tilde{x}^*(\cdot|t), \tilde{v}^*(\cdot|t))$ and define $C := \max_{x \in \mathcal{X}} \|\lambda^\top x\|$. Analogous to [21, Lemma. 3], there exists a $k_x \geq 1$, such that the optimal rotated trajectory $\tilde{x}^*(\cdot|t)$ and $\overline{x}_\epsilon(\cdot|t)$ simultaneously satisfy

$$\|\overline{x}_\epsilon(k_x|t)\|_Q^2 \leq 2\frac{\tilde{V} + 2C}{N-2} =: \sigma(N-2),$$
$$\|\tilde{x}^*(k_x|t)\|_Q^2 \leq \sigma(N-2), \quad \sigma \in \mathcal{L}.$$

For a sufficiently large N, we have $\overline{x}_\epsilon(k_x|t) \in \mathcal{X}_f$. Thus the inexact trajectory can be appended by the terminal controller K_f (see Prop. 16). This implies the following bound

$$
\begin{aligned}
&\tilde{V}_N(x(t+1)) \\
\leq &\sum_{k=1}^{k_x-1} \|\overline{x}_\epsilon(k+1|t)\|_Q^2 + \|v_\epsilon(k+1|t)\|_R^2 \\
&+ \tilde{V}_{N-k_x+1}(\overline{x}_\epsilon(k_x|t)) \\
\leq &\tilde{J}_{k_x}(\tilde{x}^*(\cdot|t), \tilde{v}^*(\cdot|t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \eta \\
&+ 2\|\lambda\| \frac{\sqrt{\sigma(N-2)}}{\lambda_{\min}(Q)} + \lambda_{\max}(P/Q)\sigma(N-2) \\
&+ \alpha_V(2\sigma(N-2)) \\
\leq &\tilde{V}_N(x(t)) - \|x(t)\|_Q^2 - \|v(t)\|_R^2 + \eta + \theta(N-2), \quad \theta \in \mathcal{L}.
\end{aligned}
$$

Recursive satisfaction of $\tilde{V}_N \leq \tilde{V} - \eta$ can be ensured for

$$\tilde{V} - \eta \geq \alpha \geq (\eta + \theta(N-2))\lambda_{\max}(P/Q).$$

The inexact trajectory satisfies the turnpike property with

$$\sigma_\epsilon(N-2) := 2\frac{\tilde{V}_\epsilon + 2C}{N-2},$$

which results in (A.11), with θ_ϵ defined analogous to θ. □

Proposition 18 shows that practical stability of the optimal steady-state can be ensured under inexact dual minimization with minimal assumptions. While the result may look rather technical, the resulting theory can be very useful for practical application. In particular, suppose we have some desirable, possibly unreachable, operation point. Then there exists a steady-state with a minimal distance to this operation point (measured in terms of the economic stage cost). In general, this optimal steady-state might not be known and thus using a terminal cost and/or constraints to stabilize this optimal steady-state might be difficult in practice. Proposition 18 provides an elegant and practical solution to this problem. Since we use no terminal cost, the optimal steady-state does not need to be known in order to implement the inexact distributed economic MPC scheme. Thus, this result ensures that it is possible to (approximately) stabilizes the (in general unknown) optimal steady-state for large-scale systems by using finite dual iterations. The application of these results to real-time economic dispatch of distributed power networks is explored in [24].

Remark 19 *For both setups (inexact distributed economic MPC with terminal cost and without terminal cost), it is possible to derive guarantees for the transient economic performance of the resulting closed-loop system, similar to the nominal guarantees in [12,14] (compare [20, Sec. 4.4]).*

A.5 Multi-step MPC

In multi-step MPC, the first $M \in \mathbb{N}$ steps of the input sequence $v_\epsilon(\cdot|t)$ are applied and the optimization problem is solved again after M time steps. In the considered setup, this can have multiple advantages, which are discussed in more detail. The following set of (less restrictive) conditions ensure strict recursive feasibility under inexact minimization by applying $M \leq N$ steps of the open-loop solution:

$$
\begin{aligned}
&\epsilon_{z,k} = \epsilon_\lambda + \frac{N-1-k}{M}(\epsilon_\lambda + \epsilon), \\
&\overline{h}_{j,k} = h_j - \epsilon_x - \sigma_{\mathcal{W}}(H_j^\top, k) - \frac{k}{M}(\epsilon_x + \epsilon_\lambda), \\
&\overline{l}_{j,k} = l_j - \epsilon_u - \sigma_{\mathcal{W}}(K^\top L_j^\top, k) - \frac{k}{M}(\epsilon_u + \epsilon_\lambda), \\
&(A+BK_f)^M(\overline{\mathcal{X}}_{f,\epsilon} \oplus_{l=0}^{M-1} A_K^{N-l-1}\mathcal{W}_l) \subseteq \overline{\mathcal{X}}_{f,\lambda}, \quad \text{(A.12)} \\
&(A+BK_f)^k(\overline{\mathcal{X}}_{f,\epsilon} \oplus_{l=0}^{M-1} A_K^{N-l-1}\mathcal{W}_l) \\
&\subseteq \{x |\ Hx \leq \overline{h}_{N-M+k} - \mathbf{1}_p\epsilon_\lambda\}, \quad k \leq M-1, \\
&K_f(A+BK_f)^k(\overline{\mathcal{X}}_{f,\epsilon} \oplus_{l=0}^{M-1} A_K^{N-l-1}\mathcal{W}_l) \\
&\subseteq \{u |\ Lx \leq \overline{l}_{N-M+k} - \mathbf{1}_q\epsilon_\lambda\}, \quad k \leq M-1.
\end{aligned}
$$

For $M = 1$ we recover the conditions in Section 3. The parameters $\epsilon_{z,k}$ and the resulting constraint tightening are significantly smaller for larger M. Furthermore, the online computation becomes significantly less demanding due to the longer sampling time in between optimizations and because a larger possible inaccuracy ϵ can be chosen (due to the less restrictive constraint tightening).

As discussed in Section A.1, designing a distributed polytopic terminal set (Ass. 1,4) can be difficult due to the required (strict) positive invariance. Condition (A.12) can be satisfied with arbitrary (compact) distributed terminal polytopic sets for a sufficiently large M and small enough ϵ (compare [17,18]).

An additional advantage of multi-step MPC is that the necessary prediction horizon N for MPC without terminal cost becomes smaller (compare [13, Sec. 10.4]).

A.6 Nonlinear system dynamics

An extension of the proposed method to nonlinear system dynamics is not straightforward. This requires an extension of the k-step support function [4] to nonlinear system dynamics. To overcome this challenge, nonlinear robust MPC approaches such as [23] might be useful to compute a simple over-approximation of the constraint tightening.

Additionally, an extension of Section 3.5 to the approximate solution of distributed nonlinear programs (NLP) with finite iterations is required, see for example [16,15].