A Convex Conic Underestimate of Laplacian Spectra and its Application to Network Synthesis

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***Abstract*— We derive sufficient conditions on the eigenvectors and eigenvalues of a matrix for letting it be the Laplacian of an undirected, weighted graph. The derived conditions are convex conic and apply to network synthesis problems. We present an algorithmic implementation of the proposed synthesis procedure.**

I. Introduction

Given an undirected, weighted graph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{W})$, i.e.

$$\begin{aligned}&\mathscr{V} = \{1 \cdots n\} \subset \mathbb{N},\\ &\mathscr{E} \subset (\mathscr{V} \times \mathscr{V}) \quad \text{symmetric},\\ &\mathscr{W} : \mathscr{E} \to (0, \infty), \quad (i,j) \mapsto w_{ij} \quad \text{symmetric},\end{aligned}$$

its symmetric, positive semidefinite Laplacian $L \in \mathbb{R}^{n\times n}$ can be defined via

$$L \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{(1,j)\in\mathscr{E}} w_{1j}\,(x_1 - x_j) \\ \vdots \\ \sum_{(n,j)\in\mathscr{E}} w_{nj}\,(x_n - x_j) \end{bmatrix} \tag{1}$$

and from the spectral theorem we know that L admits an eigendecomposition

$$L = V\Lambda V^{-1}$$

with the property that

$$\begin{aligned}V &= [v_1 \cdots v_n] \in \mathbb{R}^{n\times n} \quad \text{orthonormal},\\ \Lambda &= \operatorname{diag}(\lambda_1 \cdots \lambda_n) \in \mathbb{R}^{n\times n} \quad \text{real},\end{aligned}$$

where $\operatorname{diag}(\lambda_1 \cdots \lambda_n)$ here and henceforth denotes the diagonal matrix with diagonal entries $\lambda_1 \cdots \lambda_n$. As L is positive semidefinite, all λ_i are nonnegative. Moreover, from its very definition (1), we know that there is an $i \in \mathscr{V}$ with the property that $\lambda_i = 0$ and $\sqrt{n}v_i^\top = [1 \cdots 1] =: 1_n^\top$. Fiedler has proven (cf. [1]) that, if $\mathscr{G}$ is connected, there is no other zero eigenvalue. Without losing generality, we set this aforementioned i to 1 in the remainder.

Unfortunately, the converse is not true (cf. [1]), i.e. if we choose an orthonormal matrix V with $\sqrt{n}v_1 = 1_n$ and a real matrix Λ with $\lambda_1 = 0$ and $\lambda_2 \cdots \lambda_n > 0$, then $V\Lambda V^{-1}$ is not a Laplacian in general. In particular, $V\Lambda V^{-1}$ may have positive offdiagonal elements, which a Laplacian does not have (cf. (1)). In the following, we derive sufficient conditions on V and Λ to let $V\Lambda V^{-1}$ be a Laplacian.

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II. A Convex Conic Underestimate of Laplacian Spectra

In the previous section, we said that orthonormal V with $\sqrt{n}v_1 = 1_n$ and real Λ with $\lambda_1 = 0$ and $\lambda_2 \cdots \lambda_n > 0$ does not imply that $V\Lambda V^{-1}$ is a Laplacian. In this section, we derive additional conditions that let this implication be true.

The following proposition led to our main idea.

Proposition 1: Let V be orthonormal with $\sqrt{n}v_1 = 1_n$ and let Λ be real with $\lambda_1 = 0$. For every $c \in (0, \infty)$, if $(\lambda_2 \cdots \lambda_n) \in \{c\}^{n-1}$, then $V\Lambda V^{-1}$ is the Laplacian of a connected graph.

Proof: As V is orthonormal, we have $V^{-1} = V^\top$. Thus, with $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_n = c$, the ith entry of the jth column of $V\Lambda V^{-1}$ is given by

$$\sum_{k=1}^{n} \lambda_i v_{ki} v_{kj} = \sum_{k=2}^{n} c v_{ki} v_{kj}, \tag{2}$$

where v_{ki} denotes the ith entry of v_k. With the same reason, we have that $VV^\top = I$ with I denoting the identity of $\mathbb{R}^{n\times n}$ (such that we henceforth replace V^{-1} by $V^\top$), yielding

$$\sum_{k=1}^{n} v_{ki} v_{kj} = \delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta. Considering the case where $i \neq j$ and multiplying with c, we have

$$\sum_{k=1}^{n} c v_{ki} v_{kj} = 0. \tag{4}$$

Subtracting the latter from (2), we find that offdiagonal entries of $V\Lambda V^\top$ read $-cv_{1i}v_{1j}$, which equals $-\frac{c}{n}$ by our very choice of v_1, and which is hence a negative quantity. All other properties of a Laplacian of a connected graph follow as outlined in section I. This concludes the proof. ∎

Remarkably, by the foregoing choice of Λ, we attain strictly negative offdiagonal elements of $V\Lambda V^\top$. This leads to the following corollary.

Corollary 1: Let V be orthonormal with $\sqrt{n}v_1 = 1_n$ and let Λ be real with $\lambda_1 = 0$. For every $c \in (0, \infty)$, there exists a neighborhood U_c of $\{c\}^{n-1}$ such that for every $(\lambda_2 \cdots \lambda_n) \in U_c$, $V\Lambda V^\top$ is the Laplacian of a connected graph.

If we could thus obtain a characterization of such U_c, then

$$U := \bigcup_{c\in(0,\infty)} U_c \tag{5}$$

would be an appropriate underestimate of Laplacian spectra in the sense that $(\lambda_2 \cdots \lambda_n) \in U$ would imply that $V\Lambda V^\top$ is a Laplacian.

For the sake of such a characterization of neighborhoods U_c, define

$$U_{ci} \coloneqq \{c\}^{i-2} \times \left[c\frac{n-2}{n}, c\frac{n}{n-2}\right] \times \{c\}^{n-i} \tag{6}$$

in the following. Let conv denote the convex hull.

Lemma 1: Let V be orthonormal with $\sqrt{n}v_1 = 1_n$ and let Λ be real with $\lambda_1 = 0$. For every $c \in (0, \infty)$, for every $(\lambda_2 \cdots \lambda_n) \in \operatorname{conv}(U_{c2} \cdots U_{cn})$, $V \Lambda V^\top$ is the Laplacian of a connected graph.

Proof: By our very choice, $\lambda_2 \cdots \lambda_n$ are a convex combination from $(U_{c2} \cdots U_{cn})$. Thus, for any $i \neq 1$, it follows from the definition of U_{ci} that

$$\lambda_i = c + a_{2(i-1)-1}\frac{2c}{n-2} - a_{2(i-1)}\frac{2c}{n}, \tag{7}$$

where the coefficients a_i suffice

$$\sum_{i=1}^{2(n-1)} a_i = 1. \tag{8}$$

Substituting this and $\lambda_1 = 0$ into (2), we find that the ith entry of the jth column of $V \Lambda V^\top$ reads

$$\sum_{k=2}^{n} \left(c + a_{2(k-1)-1}\frac{2c}{n-2} - a_{2(k-1)}\frac{2c}{n}\right) v_{ki}v_{kj}. \tag{9}$$

Following the proof of Proposition 1 further, one finds that the latter equals

$$-\frac{c}{n} + \sum_{k=2}^{n} \left(a_{2(k-1)-1}\frac{2c}{n-2} - a_{2(k-1)}\frac{2c}{n}\right) v_{ki}v_{kj}. \tag{10}$$

By virtue of Lemma A3 (postponed to the appendix), which we postponed to the appendix, this is less than or equal to

$$\begin{aligned} &-\frac{c}{n} + \sum_{k=2}^{n} \left(a_{2(k-1)-1} + a_{2(k-1)}\right)\frac{c}{n} \\ &= -\frac{c}{n} + \left(\sum_{i=1}^{2(n-1)} a_i\right)\frac{c}{n} = 0. \end{aligned} \tag{11}$$

This, in turn, lets all offdiagonal elements of $V \Lambda V^\top$ be non-positive. All other properties of a Laplacian of a connected graph follow as outlined in section I. This concludes the proof. ■

Now, for any $c \in (0, \infty)$, define the polytope

$$U_c \coloneqq \operatorname{conv}(U_{c2} \cdots U_{cn}) \tag{12}$$

and U as in (5) to state our main result.

Theorem 1: Let V be orthonormal with $\sqrt{n}v_1 = 1_n$ and let Λ be real with $\lambda_1 = 0$. If $(\lambda_2 \cdots \lambda_n) \in U$, then $V \Lambda V^\top$ is the Laplacian of a connected graph. Moreover, U is a convex cone.

Proof: The first statement is a direct consequence of Lemma 1. For the second statement, it is sufficient that

$$\bigcup_{c \in \mathbb{R}} \partial U_{ci} \tag{13}$$

is a union of vector spaces and that, for all $c \in (0, \infty)$, U_c is convex. ■

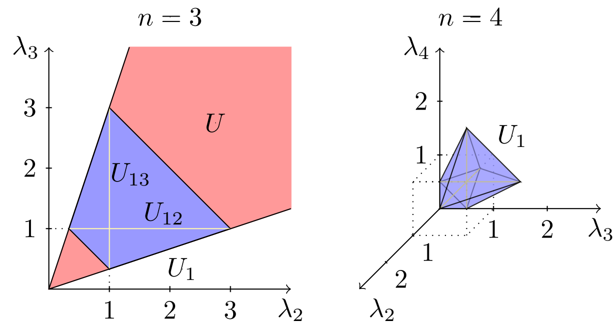

Fig. 1. Illustration of the convex polytope U_c with $c = 1$ for $n = 3$ (left: ■) and $n = 4$ (right: ■) (i.e. the convex hulls of (U_{12}, U_{13}) (left: ■) and (U_{12}, U_{13}, U_{14}) (right: ■), respectively) and the convex cone U (i.e. the union of U_c over $c \in (0, \infty)$) for $n = 3$ (left: ■)

In Fig. 1, we depict the convex polytope U_c with $c = 1$ for $n = 3$ and $n = 4$. For the former case, we have also plotted the convex cone U (this was omitted in the latter case for the sake of visibility).

III. Application to Network Synthesis

Network synthesis has been a problem of recent interest due to the increasing focus on networked control systems. These problems are usually solved using techniques from optimal control via constrained minimization of an appropriately chosen cost functional [2]–[7].

In this section, we apply our findings from the foregoing section to network synthesis problems by choosing eigenvectors and eigenvalues of its Laplacian. This is particularly appealing in diffusively coupled systems, which are usually modeled via the differential equation

$$\dot{x} = -Lx, \tag{14}$$

whose solution

$$\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad (x_0, t) \mapsto \varphi(x_0, t) \tag{15}$$

is given via

$$\begin{aligned} \varphi(x_0, t) &= \mathrm{e}^{-Lt}\, x_0 && (16) \\ &= V \operatorname{diag}\left(\mathrm{e}^{-\lambda_1 t} \cdots \mathrm{e}^{-\lambda_n t}\right) V^{-1} x_0 && (17) \\ &= \sum_{i=1}^{n} v_i\, \mathrm{e}^{-\lambda_i t}\, v_i^\top x_0 && (18) \end{aligned}$$

(where the last equality follows from orthonormality of V) and thus completely determined via its initial condition x_0 and its eigenvalue-eigenvector pairs. It is readily verified that, if $\mathscr{G}$ is connected, then all solutions approach the subspace

$$\mathscr{S} \coloneqq \operatorname{span}(1_n), \tag{19}$$

in which case the network is called a consensus network. These networks have received particular attention in recent research (cf. [8] and references therein). As Theorem 1 yields sufficient conditions for Laplacians associated with connected graphs, we will thus restrict ourselves to synthesis of consensus networks.

It follows that a direct choice of eigenvectors and eigenvalues of a Laplacian allows to directly influence the solutions of consensus networks. For instance, the rate at which φ approaches $\mathscr{S}$ is directly influenced by the magnitude of the second smallest eigenvalue of L (cf. [9]) and minimization of the spectral radius of L increases the rate at which iterates of the difference equation associated with (14) approach $\mathscr{S}$ (cf. [10]).

We are thus interested in synthesis procedures in which a (possibly incomplete, i.e. $m \leq n$) set of chosen eigenvalue-eigenvector pairs $(\mu_1, w_1) \cdots (\mu_m, w_m)$ is mapped to a set of eigenvalue-eigenvector pairs $(\lambda_1, v_1) \cdots (\lambda_n, v_n)$ such that $V \Lambda V^\top$ is the Laplacian of a connected graph (similarly to [11]). Along section I, we assume $\mu_1 = 0$ and $\sqrt{n} w_1 = 1_n$.

We propose the following 4-step synthesis procedure:

(i) Find $\mu_{m+1} \cdots \mu_n$ such that $d\left(\left(\mu_2 \cdots \mu_n\right), U\right)$ (where $d(x, U) := \inf\{\|x - u\| \,|u \in U\}$) is minimized.

(ii) Set $(\lambda_1 \cdots \lambda_n) := \mathrm{P}(\mu_1 \cdots \mu_n)$, where $\mathrm{P} : \mathbb{R}^n \to \{0\} \times U$ denotes a projection to U.

(iii) Determine appropriate $v_1 \cdots v_n$ via orthonormalization of $w_1 \cdots w_m$.

(iv) Find the Laplacian $V \Lambda V^\top$ by virtue of Theorem 1.

First, complete step (i). For doing so, let μ_0 denote

$$\mu_0 := \min\{\mu_2 \cdots \mu_m\} : \quad n \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i \leq \mu_0}} \mu_i > (n-2) \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i > \mu_0}} \mu_i \tag{20}$$

in what follows.

Lemma 2: There is no choice of $\mu_{m+1} \cdots \mu_n$ which satisfies $d\left(\left(\mu_2 \cdots \mu_n\right), U\right) < d\left(\left(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0\right), U\right)$.

Proof: We prove that $(\mu_0 \cdots \mu_0)$ minimizes

$$f : \mathbb{R}^{m-n} \to [0, \infty), \quad (\mu_{m+1} \cdots \mu_n) \mapsto d\left(\left(\mu_2 \cdots \mu_n\right), U\right). \tag{21}$$

For doing so, squash the image of f and instead consider

$$\begin{aligned} f' :& \mathbb{R}^{m-n} \times (0, \infty) \to [0, \infty), \quad \left(\left(\mu_{m+1} \cdots \mu_n\right), c\right) \mapsto \\ & \frac{n}{2c} \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i \leq c}} |c - \mu_i| + \frac{n-2}{2c} \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i > c}} |\mu_i - c| = \qquad (22) \\ & \frac{n}{2c} \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i \leq c}} (c - \mu_i) + \frac{n-2}{2c} \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i > c}} (\mu_i - c). \qquad (23) \end{aligned}$$

It is readily verified that

$$\begin{aligned} & f'_c : \mathbb{R}^{m-n} \to [0, \infty), \\ & (\mu_{m+1} \cdots \mu_n) \mapsto \inf_{c \in (0, \infty)} f'\left(\left(\mu_{m+1} \cdots \mu_n\right), c\right) \end{aligned} \tag{24}$$

increases as f increases by the very definition of U_c, but f'_c has its image squashed such that

$$\left(\left(\mu_2 \cdots \mu_n\right) \in U\right) \Leftrightarrow \left(f'_c\left(\left(\mu_{m+1} \cdots \mu_n\right), c\right) \leq 1\right). \tag{25}$$

It it thus sufficient to characterize the minima of f'_c in order to characterize the minima of f. Now, in order to infimize f' over c, define

$$f'_\mu : (0, \infty) \to [0, \infty), \quad c \mapsto f'\left(\left(\mu_{m+1} \cdots \mu_n\right), c\right). \tag{26}$$

Then f'_μ is continuous and continuously differentiable almost everywhere (except at the arguments μ_i) with $\nabla f'_\mu$ given by

$$\nabla f'_\mu(c) = \frac{1}{2c^2} \left(n \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i \leq c}} \mu_i - (n-2) \sum_{\substack{i \in \mathscr{V} \setminus \{1\} \\ \mu_i > c}} \mu_i \right). \tag{27}$$

It is readily inferred that $\nabla f'_\mu$ has the properties that

$$\left(\nabla f'_\mu(c_0) < 0\right) \Rightarrow \left(\forall c \in (0\,, c_0\,]\,, \nabla f'_\mu(c) < 0\right), \tag{28}$$

$$\left(\nabla f'_\mu(c_0) > 0\right) \Rightarrow \left(\forall c \in [c_0, \infty)\,, \nabla f'_\mu(c) > 0\right). \tag{29}$$

There thus either exists a left-closed, right-open $C \subset (0, \infty)$ or a singleton $c_0 \in (0, \infty)$ with the property that f'_μ is nowhere smaller than at the closure of C or at c_0, respectively. For the former case, set $c_0 := \sup C$ as a convention. It follows that

$$f'_c\left(\mu_{m+1} \cdots \mu_n\right) = f'\left(\left(\mu_{m+1} \cdots \mu_n\right), c_0\right) \tag{30}$$

and that hence

$$\nabla f'_c\left(\mu_{m+1} \cdots \mu_n\right) = \begin{bmatrix} \begin{cases} \frac{-n}{2c_0} & \mu_{m+1} \leq c_0 \\ \frac{n-2}{2c_0} & \mu_{m+1} > c_0 \end{cases} \\ \vdots \\ \begin{cases} \frac{-n}{2c_0} & \mu_n \leq c_0 \\ \frac{n-2}{2c_0} & \mu_n > c_0 \end{cases} \end{bmatrix} \tag{31}$$

such that f'_c can assume no smaller value than it does at $(c_0 \cdots c_0)$ (as $\nabla f'_c$ has negative entries for all smaller choices and positive entries for all larger choices), i.e. there is no choice of $\mu_{m+1} \cdots \mu_n$ which satisfies $d\left(\left(\mu_2 \cdots \mu_n\right), U\right) < d\left(\left(\mu_2 \cdots \mu_m, c_0 \cdots c_0\right), U\right)$. Getting back to $\nabla f'_\mu$, we find that c_0 is the smallest value for which $\nabla f'_\mu$ is positive. Moreover, $\nabla f'_\mu$ is discontinuous only at the arguments μ_i, such that it must be true that $c_0 \in \{\mu_2 \cdots \mu_m\}$. Sending $\mu_{m+1} \cdots \mu_n$ to c_0 in f', we find that, for this choice, $\nabla f'_\mu$ is positive as long as

$$n \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i \leq c_0}} \mu_i > (n-2) \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i > c_0}} \mu_i \tag{32}$$

and that the smallest c_0 which satisfies the latter as well as $c_0 \in \{\mu_2 \cdots \mu_m\}$ will be just μ_0, which yields $c_0 = \mu_0$ and thus proves the claim. ■

Please note that it is also possible to choose $c_0 := \min C$ instead of $c_0 := \sup C$, for which case we would have to let μ_0 be

$$\mu_0 := \max\{\mu_2 \cdots \mu_m\} : \quad n \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i \leq \mu_0}} \mu_i \leq (n-2) \sum_{\substack{i \in \{2 \cdots m\}: \\ \mu_i > \mu_0}} \mu_i. \tag{33}$$

As proven, this would however not change the value that $d\left(\left(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0\right), U\right)$ assumes.

The foregoing lemma and its proof directly lead to the following corollary.

Corollary 2: If $(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0) \notin U$, then there exists no choice of $\mu_{m+1} \cdots \mu_n$ such that $(\mu_2 \cdots \mu_n) \in U$. If $(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0) \in U$, then $(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0) \in U_{\mu_0}$.

As there is no doubt that there are choices of $\mu_2 \cdots \mu_m$ for which $(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0) \notin U$, this necessitates step (ii), i.e. projection of $(\mu_2 \cdots \mu_m, \mu_0 \cdots \mu_0)$ to U. As stated in the corollary, we have already found a compact set, namely U_{μ_0}, which is suited for projecting to. As U_{μ_0} is a polytope, this can be solved using `simplex`. An appropriate distance to minimize in the projection would be the distance induced by the 1-norm, as this would alter the choice of $\mu_2 \cdots \mu_m$ as little as possible. Thus, let P be defined by

$$\begin{aligned} \mathrm{P} :&\mathbb{R}^n \to U, \quad (\mu_1 \cdots \mu_n) \mapsto \\ &\Big(\mu_1, \operatorname*{argmin}_{(\lambda_2 \cdots \lambda_n) \in U_{\mu_0}} \left\| (\mu_2 \cdots \mu_n) - (\lambda_2 \cdots \lambda_n) \right\|_1 \Big). \end{aligned} \tag{34}$$

This lets Λ suffice the sufficient conditions from Theorem 1 for $V \Lambda V^\top$ to be the Laplacian of a connected graph. However, an appropriate V has not yet been found, bringing us to step (iii).

In this step, we must form an orthogonal basis $v_1 \cdots v_n$ out of the given $W := [w_1 \cdots w_m] \in \mathbb{R}^{n \times m}$. For $v_1 \cdots v_m$, this can be done via the Gram-Schmidt process, i.e.

$$\begin{aligned} g_i^m :&\mathbb{R}^{n \times m} \to \mathbb{R}^n, \\ &W \mapsto w_i - \sum_{j=1}^{i-1} \frac{g_j(W) \cdot w_i}{g_j(W) \cdot g_j(W)} g_j(W). \end{aligned} \tag{35}$$

Notably, we did not insist that $w_1 \cdots w_m$ are linearly independent, such that there might be zero $g_i^m(W)$. For coping with this, and for filling the remaining $v_{m+1} \cdots v_n$, set

$$(w_{m+1} \cdots w_{m+n}) := (e_1 \cdots e_n), \tag{36}$$

where e_i denote the unit vectors of $\mathbb{R}^n$. Defining

$$W' := [w_1 \cdots w_{m+n}] = [w_1 \cdots w_m, e_1 \cdots e_n] \in \mathbb{R}^{n \times m+n}, \tag{37}$$

we can hence also compute $g_1^{m+n}(W') \cdots g_{m+n}^{m+n}(W')$ from the Gram-Schmidt process, of whom exactly m vectors will be zero. This follows from the facts that the Gram-Schmidt process returns zero vectors for linearly dependent arguments and mutually orthogonal vectors else whilst there can be no more than n mutually orthogonal vectors in $\mathbb{R}^n$.

This brings us in the position to complete step (iv), whom we state as the following theorem.

Theorem 2: Let $m \leq n$ eigenvalue-eigenvector pairs $(\mu_1, w_1) \cdots (\mu_m, w_m)$ be given with the properties $\sqrt{n} w_1 = 1_n$ and $\mu_1 = 0$. If $(\lambda_1 \cdots \lambda_n) = \mathrm{P}(\mu_1 \cdots \mu_m, \mu_0 \cdots \mu_0)$ and $v_1 \cdots v_n$ are the (exactly) n normalized nonzero members of $g_1^{m+n}(W') \cdots g_{m+n}^{m+n}(W')$, then $V \Lambda V^\top$ is the Laplacian of a connected graph. Moreover, if there exists a choice of $\lambda_{m+1} \cdots \lambda_n$ such that $(\lambda_1 \cdots \lambda_m) = (\mu_1 \cdots \mu_m)$ and such that $(\lambda_2 \cdots \lambda_n) \in U$, then the latter choice has this property. If there is no such choice, then the distance from $(\lambda_1 \cdots \lambda_m)$ to $(\mu_1 \cdots \mu_m)$ induced by the 1-norm is minimized subject to $(\lambda_2 \cdots \lambda_n) \in U$.

Proof: By the very definition of P, $(\lambda_2 \cdots \lambda_n)$ is in U and $\lambda_1 = \mu_1$. The Gram-Schmidt process has the property that $g_1^{m+n}(W') = w_1$ and that the n nonzero members of $g_1^{m+n}(W') \cdots g_{m+n}^{m+n}(W')$ (there are exactly n such members due to the argumentation just before the theorem) are mutually orthogonal. Thus, V is orthonormal. As $\mu_1 = 0$ and $\sqrt{n} w_1 = 1_n$, $V \Lambda V^\top$ is the Laplacian of a connected graph by virtue of Theorem 1. The second statement is the consequence of Lemma 2 and the last statement follows from the definition of P. This concludes the proof. ∎

IV. Algorithmic Implementation

The synthesis procedure presented in the foregoing section employed `simplex` as well as the Gram-Schmidt procedure and is thus designated to be applied in an algorithmic fashion. Also, an algorithmic implementation of the procedure enables more efficient and automated application of the proposed synthesis. We call the corresponding algorithmic implementation `eignetsyn` (short for *eig*envalue-eigenvector-based *net*work *syn*thesis) and presents its pseudocode in Algorithm 1, but its implementation in MATLAB is available online at `www.ist.uni-stuttgart.de/~montenbruck`.

Algorithm 1 (`eignetsyn`)**:**

Require: $(\mu_2, w_2) \cdots (\mu_m, w_m)$ such that $\mu_2 < \cdots < \mu_m$
1: $(\mu_1, \sqrt{n} w_1) \leftarrow (0, 1_n)$
2: $i \leftarrow m$
3: **repeat**
4: $\quad \mu_0 \leftarrow \mu_i$
5: $\quad i \leftarrow i - 1$
6: **until** $\neg$(20)
7: $(\mu_{m+1} \ldots \mu_n) \leftarrow (\mu_{i+2} \cdots \mu_{i+2})$
8: $(\lambda_1 \cdots \lambda_n) \leftarrow \mathrm{P}(\mu_1 \cdots \mu_n)$ solving (34) with `simplex`
9: $(w_{m+1} \cdots w_{m+n}) \leftarrow (e_1 \cdots e_n)$
10: $j \leftarrow 1$
11: **for all** $k \in \{1 \cdots m+n\}$ **do**
12: $\quad$ **if** $g_k^{m+n}([w_1 \cdots w_{m+n}]) \neq 0$ **then**
13: $\quad\quad v_j \leftarrow$ normalized $g_k^{m+n}([w_1 \cdots w_{m+n}])$
14: $\quad\quad j \leftarrow j + 1$
15: $\quad$ **end if**
16: **end for**
17: $V \leftarrow [v_1 \cdots v_n]$
18: $\Lambda \leftarrow \operatorname{diag}(\lambda_1 \cdots \lambda_n)$

Return: $V \Lambda V^\top$

Algorithm 1 follows the steps (i)-(iv) from the foregoing section whilst completing (i) in lines 1-7, (ii) in line 8, (iii) in lines 9-16, and (iv) in lines 17-18. The algorithm requires the eigenvalues $\mu_2 \cdots \mu_m$ to be sorted but one could also implement a sorting procedure in its very beginning instead. In line 8, we explicitly refer to `simplex`, but in lines 12-13 we implicitly refer to the Gram-Schmidt process (35).

We wish to remark that a graph is uniquely determined by its Laplacian and that one could thus also directly return the graph instead of its Laplacian.

V. Numerical Examples

We present two examples, of whom the first example illustrates step (ii) whilst the second example illustrates step (iii).

In our first example, which illustrates step (ii), we highlight one fact that we did not discuss before. Namely, in many situations, the orthogonal affine subspace which $\mu_{m+1} \cdots \mu_n$ live on intersects with the convex cone U. In such a situation, one could thus omit the projection P and directly set $(\lambda_1 \cdots \lambda_n) \leftarrow (\mu_1 \cdots \mu_m, \mu_0 \cdots \mu_0)$ after having computed μ_0. For instance, pick $m \leftarrow 3$ and $n \leftarrow 4$ with

$$(\mu_1 \cdots \mu_3) \leftarrow (0, 1, 2)\,. \tag{38}$$

For $\mu_0 \leftarrow \mu_2$, $\neg$(20) is true, such that we set $\mu_0 \leftarrow \mu_3$ and

$$(\mu_n = \mu_{m+1} \leftarrow 2) \wedge ((\lambda_1 \cdots \lambda_n) \leftarrow (0, 1, 2, 2))\,. \tag{39}$$

We thus expect that $(\lambda_2 \cdots \lambda_n) \in U_2$ for this choice. To check this, compute

$$U_{22} = [1, 4] \times \{2\}^2 \tag{40}$$

$$U_{23} = \{2\} \times [1, 4] \times \{2\} \tag{41}$$

$$U_{24} = \{2\}^2 \times [1, 4] \tag{42}$$

and find that, indeed,

$$(\lambda_2 \cdots \lambda_n) \in \partial U_{22} \subset U_{22} \subset U_2 \subset U. \tag{43}$$

It remains to check whether or not one can compute the Laplacian of a connected graph from this. For doing so, choose $\sqrt{4}v_1 \leftarrow 1_4$ and

$$\sqrt{2}v_2 \leftarrow \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \sqrt{2}v_3 \leftarrow \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \sqrt{4}v_4 \leftarrow \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \tag{44}$$

such that V is orthonormal and

$$VDV^\top = \frac{1}{2}\begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \tag{45}$$

is the Laplacian of a connected graph, as expected. Also, in this example, the condition provided by U is tight, as $(1, 2, 2) \in \partial U_{22}$ and for any choice of λ_4 which is greater than 2, the second entry of the first row (and the first entry of the second row) of $VDV^\top$ becomes positive, such that $VDV^\top$ is no Laplacian for such a choice. We illustrate how the orthogonal affine subspace which λ_4 lives on intersects with U_2 in Figure 2. In the figure, one also observes that this affine subspace intersects with a vertex of U_2. More general, it is not possible that affine subspaces which $\mu_{m=1} \cdots \mu_n$ live on contain edges or faces of U_2, as the mentioned affine subspaces are by definition orthogonal whilst the edges and faces of any U_c cannot be orthogonal; of course, it is still possible that the affine subspace intersects with edges or faces of U_2. Another information that we take out of this interpretation is that for $m = 2$, the affine subspace must intersect with U due to its dimension and orthogonality.

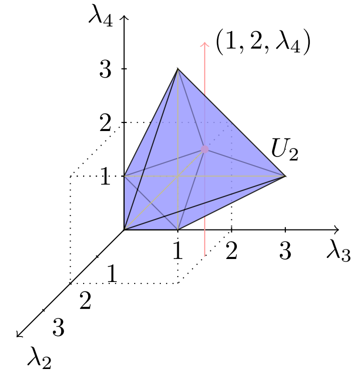

Fig. 2. Illustration of the convex polytope U_c with $c = \mu_0 = 2$ for $n = 4$ ( ) (i.e. the convex hull of (U_{22}, U_{23}, U_{24}) ( ) from (40)-(42)), the orthogonal affine subspace which $\mu_{m+1} = \mu_n = \lambda_n$ lives on ( ) determined by (38), and their intersection point ( ) given by (1,2,2)

Our second example illustrates step (iii), i.e. the projection P to U via minimization of the distance induced by the 1-norm whilst satisfying the constraints defined by the polytope U_{μ_0}, such that it is possible to compute P using `simplex`. For instance, pick $m = n = 3$ with

$$(\mu_1 \cdots \mu_3) \leftarrow \left(0, \frac{3}{2}, 6\right). \tag{46}$$

More, choose $\sqrt{3}v_1 \leftarrow 1_3$ and

$$\sqrt{2}v_2 \leftarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \sqrt{6}v_3 \leftarrow \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{47}$$

such that V is orthonormal. Nevertheless,

$$V \operatorname{diag}(\mu_1, \mu_2, \mu_3)\, V^\top = \frac{1}{4}\begin{bmatrix} 16 & -8 & -8 \\ -8 & 7 & 1 \\ -8 & 1 & 7 \end{bmatrix} \tag{48}$$

is not a Laplacian. Thus, compute μ_0 by trying $\mu_0 \leftarrow \mu_2$ and finding that $\neg$(20) is true for this choice such that $\mu_0 \leftarrow \mu_3$ is to be chosen. Next, apply the projection P yielding

$$(\lambda_1 \cdots \lambda_3) \leftarrow \mathrm{P}\left(0, \frac{3}{2}, 6\right) = (0, 2, 6) \in U_{\mu_0} = U_6, \tag{49}$$

which indeed satisfies our specification that we wanted to alter the given eigenvalues $\mu_1 \cdots \mu_m$ as little as possible (in particular, we have $\mu_3 = \lambda_3$). Also, as expected,

$$VDV^\top = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} \tag{50}$$

is the Laplacian of a connected graph. The original data (μ_2, μ_3) as well as its projection and the polytope U_{μ_0} are depicted in Figure 3. Again, the condition provided by U is sharp, as the projection P has sent the positive elements of $V \operatorname{diag}(\mu_1, \mu_2, \mu_3)\, V^\top$ (i.e. the third entry of the second row / the second entry of the third row) to zero in $VDV^\top$, and not to something negative.

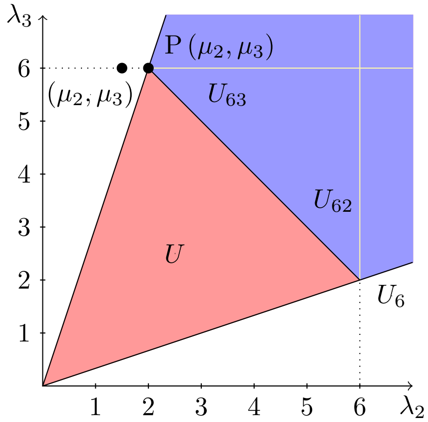

Fig. 3. Illustration of the convex polytope U_c with $c = \mu_0 = 6$ for $n = 3$ (■) (i.e. the convex hull of (U_{62}, U_{63}) (■)) and the convex cone U (■) with (μ_2, μ_3) (●) as in (46) and its projection P (μ_2, μ_3) (●) from (49) solved by minimizing the distance from (μ_2, μ_3) to U_{μ_0} induced by the 1-norm (as in (34)) with `simplex`

VI. Conclusion

We derived a convex conic subset of the spectra of Laplacians of connected, undirected, weighted graphs and applied this characterization to network synthesis problems in which specific modes of diffusively coupled systems are desired. We not only presented an algorithmic implementation of the proposed synthesis procedure, but also tested it on numerical examples, in which the proposed characterization turned out to be sharp.

Appendix

Lemma A3: If $v_k = [v_{k1} \cdots v_{kn}] \in \mathbb{R}^n$ is normalized and $v_k \cdot 1_n = 0$, then, for any distinct i, $j \in \mathscr{V}$, $v_{ki}v_{kj} \in \left[-\frac{1}{2}, \frac{n-2}{2n}\right]$.

Proof: We prove the statement separately for $n = 2$ and for $n \geq 3$.

For $n = 2$, v_k is unique up to its sign. For both possible choices, $v_{k1}v_{k2} = -\frac{1}{2}$.

For $n \geq 3$, set i to 1 and j to 2 w.l.o.g. and define

$$h : \mathbb{R}^n \to \mathbb{R}, \quad v_k \mapsto v_{k1}v_{k2}. \tag{51}$$

Now choose an orthonormal $T = [t_1 \cdots t_n] \in \mathbb{R}^{n \times n}$ with $\sqrt{n}t_1 = 1_n$. Such T is unique up to order of columns, e.g.

$$t_2^\top = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \end{bmatrix}, \tag{52}$$

$$t_3^\top = \sqrt{\frac{n-2}{2n}} \begin{bmatrix} 1 & 1 & \frac{2}{2-n} & \cdots & \frac{2}{2-n} \end{bmatrix} \tag{53}$$

and $t_4 \cdots t_n$ having their first two entries 0. With this T, perform the change of coordinates

$$v_k \mapsto T^\top v_k =: z = [z_1 \cdots z_n]. \tag{54}$$

In these coordinates, h reads

$$h(v_k) = \frac{1}{n}z_1^2 - \frac{1}{2}z_2^2 + \frac{n-2}{2n}z_3^2 + \sqrt{\frac{2(n-2)}{n^2}}z_1z_3 \tag{55}$$

with the constraints that $z_1 = 0$ (as $v \cdot 1_n = 0$) and that z is normalized (as (54) is distance preserving).

Due to this last constraint, it is possible to further perform the change of coordinates

$$\mathbb{R}^n \to \mathbb{S}^{n-1}, \quad z \mapsto (\alpha, \beta, \gamma, \cdots), \tag{56}$$

where $\mathbb{S}^{n-1}$ denotes the unit $(n-1)$-sphere, letting h read

$$h(v_k) = \begin{cases} -\frac{1}{2}\sin(\beta)^2 + \frac{n-2}{2n}\cos(\beta)^2 & n = 3 \\ -\frac{1}{2}\sin(\beta)^2 + \frac{n-2}{2n}\cos(\beta)^2\sin(\gamma)^2 & n > 3 \end{cases} \tag{57}$$

where we substituted $\alpha = 0$ as $z_1 = 0$. We implicitly define

$$h' : \mathbb{R}^2 \to \mathbb{R}, \quad (\beta, \gamma) \mapsto h(v_k) \tag{58}$$

through the foregoing equation, yielding

$$\nabla h'(\beta,\gamma) = \begin{cases} \begin{bmatrix} \frac{4}{3}\sin(\beta)\cos(\beta) \\ 0 \end{bmatrix} & n = 3 \\ \begin{bmatrix} -\sin(\beta)\cos(\beta) - \frac{n-2}{n}\cos(\beta)\sin(\beta)\sin(\gamma)^2 \\ \frac{n-2}{n}\sin(\gamma)\cos(\gamma)\cos(\beta)^2 \end{bmatrix} & n > 3 \end{cases} \tag{59}$$

to now solve $\nabla h'(\beta, \gamma) = 0$. In doing so, it is enough to restrict ourselves to $\beta \in [0, 2\pi]$ for $n = 3$ and to $\beta \in [0, \pi]$, $\gamma \in [0, 2\pi]$ in order to cover any value on $\mathbb{S}^{n-1}$, as the latter is compact. We arrive at

$$\nabla h'(\beta,\gamma) = 0 \Leftrightarrow \begin{cases} \beta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\} & n = 3 \\ \beta = \frac{\pi}{2} \vee \left(\beta = 0 \wedge \gamma \in \{0, \frac{\pi}{2}, \pi\}\right) & n > 3 \end{cases} \tag{60}$$

and find that h' attains values in $\left[-\frac{1}{2}, \frac{n-2}{2n}\right]$ for any of these solutions to $\nabla h' = 0$, where the latter is a necessary condition for extrema of h'. As h' is continuous and $\mathbb{S}^{n-1}$ is compact, h' attains its extrema on $\mathbb{S}^{n-1}$ due to the extreme value theorem. Consequently, h' is confined to $\left[-\frac{1}{2}, \frac{n-2}{2n}\right]$. As we had insisted that $h' = h$, this concludes the proof. ■

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