Extremum Seeking and Obstacle Avoidance on the Special Orthogonal Group

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Abstract: We are motivated by the idea of finding feedback laws for rotations that only require transmission of scalar signals. In particular, we develop an extremum seeking control law for systems living on the special orthogonal group. In addition, we equip our system with the capability of obstacle avoidance using the idea of navigation functions, along the lines of Koditschek and Rimon [1990].

1. INTRODUCTION

1.1 Motivating Scenario

Assume a scenario in which a satellite is equipped with a telescope and the goal is to navigate the satellite to a reference configuration R^* of its state space, which are the rotation matrices. In doing so, one is restricted to sparse communication; Specifically, it is only possible to transmit scalar-valued signals ξ. In addition, the satellite must avoid certain orientations O_i as for instance the telescope shall not face the direction of the sun. Therefore, we are interested in finding state feedback laws based only on scalar information that drive a dynamical system on the rotation matrices from an initial rotation to a target rotation whilst avoiding certain "obstacle" rotations.

1.2 Previous Work

To approach the problem described in subsection 1.1, we employ two methodologies – The path planning problem is addressed using the principles of navigation functions. The feedback law based on scalar information can be derived by means of extremum seeking.

When we say path planning problem, we mean what Kavraki and LaValle [2008] call the piano mover's problem, i.e. finding a continuous path from initial to target configuration avoiding obstacles and not leaving the workspace. The problem was initially posed by Reif [1979] in a yet different fashion. One solution to the problem that has also significantly motivated our study is to employ a navigation function, which has been proposed by Koditschek [1987]. The idea was formalized for so-called sphere worlds by Koditschek and Rimon [1990]. A problem closely related to the one presented herein was discussed by Rimon [1991]. Dürr et al. [2013a] solved a path planning problem using scalar feedback by employing extremum seeking. Global stability properties of extremum seeking in Euclidean space was proven by Tan et al. [2006] and Dürr et al. [2013a]. For a general introduction to extremum seeking systems, we refer to Ariyur and Krstić [2003]. The above results do not apply to SO(3) due to its non-Euclidean structure. We will outline this on an example.

For the solution to the problem posed in subsection 1.1, it remains to construct a dynamical system (i) that evolves on the rotation matrices, (ii) that avoids obstacles in the sense of a path planning problem and (iii) that (practically) stabilizes the destination point, i.e. it incorporates feedback into the path planning problem in the sense of Koditschek.

1.3 Contribution and Structure of the Paper

We are going to use the distance function of $\mathbb{R}^{3\times 3}$ to measure the distance to the target configuration as well as the distance to the obstacles. From this function, we will construct a navigation function in $\mathbb{R}^{3\times 3}$. Thereafter, we restrict the resulting system to SO(3). To be able to rely only on scalar signals, we construct an extremum seeking system whose solution stays "close" to the solution of the gradient system for the navigation function.

The remainder of the paper is structured as follows; In section 2, we formalize our setup and state the problem that we are going to solve. Section 3 introduces some basic methods and notions that are relevant for this work. Therein, subsection 3.1 introduces basic facts on navigation functions, subsection 3.2 contains stability definitions, and subsection 3.3 explains essential facts on extremum seeking systems. We present our main result in section 4, where we elaborate the stability properties of the proposed solution. An additional result regarding the proposed navigation function has been moved to Appendix A. We validate our ideas on a numerical example in section 5 and conclude the paper with section 6.

1.4 Notation

Rotation matrices are members of the special orthogonal group SO(3). We smoothly embed SO(3) into the $\mathbb{R}^{3\times 3}$ matrices, i.e. SO(3)$=\{R\in\mathbb{R}^{3\times 3}|R^\top R=I,\det R=1\}$, where I is the identity of $\mathbb{R}^{3\times 3}$. For the $\mathbb{R}^{3\times 3}$ matrices, we take the standard scalar product $x\cdot y=\operatorname{tr}\left(x^\top y\right)$, x, $y\in\mathbb{R}^{3\times 3}$. For the matrices forming an orthogonal basis for $\mathbb{R}^{3\times 3}$, we write E_{ij}, i.e. E_{ij} is the matrix of zeros with the ith element of the jth column set to 1. The tangent space of SO(3) at a point $R\in$SO(3) is given by T_RSO(3)$=\{R\Omega|R\in$ SO(3)$,\Omega^\top=-\Omega\}$. The Lie algebra $\mathfrak{so}(3)$ of SO(3) is canonically determined by $\mathfrak{so}(3)=\mathrm{T}_I$SO(3) and we will denote the infinitesimal generators of the algebra by Ω_i, $i=1,2,3$. The tangent space T_RSO(3) is a vector space, and we will refer to its orthogonal complement as $\mathrm{T}_R^\perp$SO(3),

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i.e. $\operatorname{tr}\left(x^\top y\right) = 0$ when $x \in \mathrm{T}_R\mathrm{SO(3)}$ and $y \in \mathrm{T}_R^\perp\mathrm{SO(3)}$. As both are linear spaces, we can define projections P from $\mathbb{R}^{3\times 3}$ to $\mathrm{T}_R\mathrm{SO(3)}$ and $\mathrm{T}_R^\perp\mathrm{SO(3)}$; We will refer to these projections as P_R and $\mathrm{P}_R^\perp$, respectively. If f is a differentiable function mapping from SO(3) to $\mathbb{R}$, by $\operatorname{grad} f$, we mean the unique vector field satisfying $\frac{\mathrm{d}}{\mathrm{d}s}\left(f \circ A\right)(s)\,|_{s=0} = \langle \operatorname{grad} f(R), V\rangle$, where $\langle\cdot,\cdot\rangle$ is the Riemannian metric, $A : [-\varepsilon,\varepsilon] \to \mathrm{SO(3)}$, $A(0) = R$, and $\frac{\mathrm{d}}{\mathrm{d}s}A(s) = V$ for every $V \in \mathrm{T}_R\mathrm{SO(3)}$. Correspondingly, by the operator Hess, we mean the form $\operatorname{Hess} f(R)(V) = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\left(f \circ A\right)(s)\,|_{s=0}$. If in contrast, g is a function mapping from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}$, we will replace grad by ∇ such that $\nabla g(x)$ denotes a matrix which has $\frac{\partial}{\partial x_{ij}} g(x)$ as the ith element of its jth column, where x_{ij} denotes the ith component of the jth column of x. Further, we replace Hess by ∇^2, such that $\nabla^2 g(x)(y)$ denotes a matrix which has $\nabla\frac{\partial}{\partial x_{ij}} g(x)\cdot y$ as the ith element of its jth column. If M is a set, then we will denote the boundary of M by ∂M and the interior of M by $\operatorname{int}(M)$. If M is a subset of SO(3), with $\operatorname{crit}_M(f)$, we refer to the set of points in M where $\operatorname{grad} f$ vanishes. If, in contrast, M is a subset of $\mathbb{R}^{3\times 3}$, with $\operatorname{crit}_M(g)$, we refer to the set of points in M where ∇g vanishes. In a differential equation $\dot{x} = f(x)$, the overdot abbreviates $\frac{\mathrm{d}}{\mathrm{d}t}$, where t is the time. In the right-hand side of the differential equation, we will often drop the explicit dependence on time whenever it can be inferred from the context. By the function $d : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \to \mathbb{R}$, we mean the distance function in $\mathbb{R}^{3\times 3}$. If we fix one of the arguments, we write $d(x,y) = d_x(y)$, such that

$$d_{R_1}(R_2) = \operatorname{tr}\left(\left(R_2 - R_1\right)^\top \left(R_2 - R_1\right)\right). \tag{1}$$

Correspondingly, we refer to an open ball in $\mathbb{R}^{3\times 3}$ by

$$\tilde{B}_{R_1}^r = \{R_2 \in \mathbb{R}^{3\times 3} | d_{R_1}(R_2) \in [0,r)\}. \tag{2}$$

If we want to exclude the members of $\mathbb{R}^{3\times 3}$ that are not members of SO(3), then we write

$$B_{R_1}^r = \{R_2 \in SO(3)\,| d_{R_1}(R_2) \in [0,r)\}. \tag{3}$$

With $\oplus$, we refer to the direct sum of vector spaces. We denote the Weingarten map of SO(3) at a point R by $\mathfrak{A}_R$. When $x \in \mathbb{R}^n$, we write $x = [x_i]$ to indicate that we denote the ith entry of x by x_i. Equivalently, when $x \in \mathbb{R}^{n\times m}$, we write $x = [x_{ij}]$ to indicate that we denote the ith entry of the jth column by x_{ij}. $\mathbb{Q}$ are the rational numbers.

2. PROBLEM STATEMENT

We are interested in controlling systems of the form

$$\dot{R} = RU, \;\; R(t_0) = R_0, \tag{4}$$

where $R_0 \in \mathrm{SO(3)}$ is the state and $U \in \mathfrak{so}(3)$ the input of the system. Our goal is to steer the system to a target configuration R^* by means of appropriate choice of U. In doing so, we restrict ourselves to feedbacks $U : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathfrak{so}(3)$ depending only on scalar-valued information, i.e.

$$U = U(\omega, \xi(R), t) \tag{5}$$

with $\xi : \mathrm{SO(3)} \to \mathbb{R}$, where $\omega = [\omega_i] \in \mathbb{R}^3$ are design parameters with $\omega_i = \alpha_i \omega$, $\alpha_i \in \mathbb{Q}$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. The latter restriction is rather technical and for details on this we refer to Dürr et al. [2013b]. The state R and the target configuration R^* are encoded in ξ, but are not available to the system.

In the course of the navigation to the attitude R^*, we restrict ourselves to a certain workspace W describing the feasible configurations, i.e. we insist that W is invariant with respect to (4) under (5). We assume that W is given in form

$$W = \{R \in \mathrm{SO}(3)\,| d_I(R) \in \left[0, r_0^2\right]\}. \tag{6}$$

(6) is commonly referred to as a sphere world (see Koditschek and Rimon [1990]). In addition, we want to avoid certain obstacles O_i whilst moving towards R^*, i.e. we insist that all O_i are repelling with some regard. We assume that there are m obstacles given by

$$O_i = \{R \in \mathrm{SO}(3)\,| d_{R_i}(R) \in \left[0, r_i^2\right]\}, \;\; i = 1,\cdots,m. \tag{7}$$

That is, either the obstacles are merely points R_i and we add the radius r_i for the sake of conservativity, or the obstacles indeed occupy volume in state space. The latter two equations restrict our maneuvering to the so-called free space

$$S = W \setminus \bigcup_{i=1}^{m} O_i, \tag{8}$$

and we assume that all obstacles are contained in the workspace and that they do not intersect, i.e. $O_i \subset W$ for all $i = 1,\cdots,m$ and $O_i \cap O_j = \emptyset$ for all $i,\ j = 1,\cdots,m$, $i \neq j$, respectively. We also presume $R^* \in \operatorname{int}(S)$. Formally, our design goal is to choose (5) such that

$$R(t) \in \operatorname{int}(S) \tag{9}$$

for all times.

3. METHODS AND PRELIMINARIES

Our main result is based on two concepts, one of which is the concept of a navigation function, i.e. a function that has the property that its gradient flows converge to R^* from almost all initial conditions without leaving S. Consequently, we include some definitions from stability theory needed in the remainder of this paper. The other concept is extremum seeking, which can be utilized to generate scalar-valued feedback laws. In particular, we will utilize an extremum-seeking approximation based on the Lie-bracket system proposed by Dürr et al. [2013b].

3.1 Navigation Functions

The concept of a navigation function was introduced by Koditschek and Rimon [1990]. The goal is to design a function which has a gradient flow converging to R^* from almost all initial conditions without leaving S.

Definition 1. (Koditschek and Rimon [1990]). Let M be compact, connected, analytic, and have nonempty boundary. An analytic, polar, Morse, admissible function $M \to \mathbb{R}$ is said to be a navigation function on M.

With this in mind, define the functions $\beta_i : \mathbb{R}^{3\times 3} \to \mathbb{R}$,

$$\beta_i(R) = \begin{cases} r_0^2 - d_I(R) & i = 0 \\ d_{R_i}(R) - r_i^2 & i = 1,\cdots,m. \end{cases} \tag{10}$$

These functions have the property that

$$S = \{R \in \mathrm{SO}(3)\,| \beta_i(R) \geq 0 \forall i = 0,\cdots,m\}. \tag{11}$$

With this relation at hand, we can also find that

$$\partial S = \{R \in \mathrm{SO}(3)\,| \exists i : \beta_i(R) = 0\}. \tag{12}$$

For the distance to the reference, we introduce $\beta^* : \mathbb{R}^{3\times 3} \to \mathbb{R}$,

$$\beta^*(R) = d_{R^*}(R), \tag{13}$$

and $\beta : \mathbb{R}^{3\times 3} \to \mathbb{R}$,

$$\beta(R) = \prod_{i=0}^{m} \beta_i(R). \tag{14}$$

In particular, we will employ the function $\varphi : \mathbb{R}^{3\times3} \to \mathbb{R}$,

$$\varphi(R) = \frac{\beta^*(R)}{\left((\beta^*(R))^k + \beta(R)\right)^{1/k}} \tag{15}$$

with sufficiently large integer k. This function has some particularly nice properties. Amongst them, φ has the image $\varphi(S) = [0,1]$ and the preimages $\varphi^{-1}(0) = R^*$ and $\varphi^{-1}(1) = \partial S$ on its extrema. Moreover, its level sets $S_h = \{R \in S | \varphi(R) \in [0,h]\}$ are compact for all $h \in (0,1]$ and we have $S_1 = S$.

Koditschek and Rimon [1990] use the vector field $-\nabla\varphi$ to employ its integral curves as solutions to the path planning problem. For application of this methodology, we need to have appropriate counterparts of W, O_i, and S in the ambient space $\mathbb{R}^{3\times3}$. Therefore, we define the ambient workspace

$$\tilde{W} = \{R \in \mathbb{R}^{3\times3} | d_I(R) \in [0, r_0^2]\}, \tag{16}$$

the ambient obstacles

$$\tilde{O}_i = \{R \in \mathbb{R}^{3\times3} | d_{R_i}(R)\,[0, r_i^2]\}, \;\; i = 1, \cdots, m, \tag{17}$$

and hence the ambient sphere world

$$\tilde{S} = \tilde{W} \setminus \bigcup_{i=1}^{m} \tilde{O}_i. \tag{18}$$

In this spirit, we also introduce the ambient system

$$\dot{\tilde{R}} = \tilde{U}, \;\; \tilde{R}(t_0) = \tilde{R}_0, \tag{19}$$

where $\tilde{R}_0 \in \mathbb{R}^{3\times3}$ is the state and $\tilde{U} \in \mathbb{R}^{3\times3}$ is the input of the system. The goal is to steer the system to the target configuration R^* by means of appropriate choice of $\tilde{U}$. In doing so, we restrict ourselves to scalar feedback laws $\tilde{U} : \mathbb{R}^{3\times3} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{3\times3}$, i.e.

$$\tilde{U} = \tilde{U}\left(\tilde{\omega}, \tilde{\xi}(\tilde{R}), t\right) \tag{20}$$

with $\tilde{\xi} : \mathbb{R}^{3\times3} \to \mathbb{R}$, where $\tilde{\omega} = [\tilde{\omega}_{ij}] \in \mathbb{R}^{3\times3}$ are design parameters with $\omega_{ij} = \alpha_{ij}\omega$, $\alpha_{ij} \in \mathbb{Q}$ and $\alpha_{ij} \neq \alpha_{kl}$ for $i \neq k$ and $j \neq l$. The latter restriction in rather technical and for details on this we refer to Dürr et al. [2013b]. The state $\tilde{R}$ and the target configuration R^* are encoded in $\tilde{\xi}$, but are not available to the system.

An important result of Koditschek and Rimon [1990] is that, by construction, φ is a navigation function on $\tilde{S}$.

Lemma 1. (Koditschek and Rimon [1990]). φ is a navigation function on $\tilde{S}$ for sufficiently large integer k.

By

$$\mathrm{crit}_{\tilde{S}}(\varphi) = \{R \in \tilde{S} | \nabla\varphi(R) = 0\}, \tag{21}$$

we denote the critical points of φ on the domain $\tilde{S}$ and by

$$\mathrm{crit}_S(\varphi) = \{R \in S | \operatorname{grad}\varphi(R) = 0\}, \tag{22}$$

we denote the critical points of φ on the domain S. As φ is Morse and $\tilde{S}$ compact, $\mathrm{crit}_{\tilde{S}}(\varphi)$ consists of isolated points and has finite cardinality. Koditschek and Rimon [1990] proposed the gradient system

$$\dot{\tilde{Z}} = -\nabla\varphi(\tilde{Z}), \;\; \tilde{Z}(t_0) = \tilde{Z}_0, \tag{23}$$

and investigated its stability properties.

Lemma 2. (Koditschek and Rimon [1990]). $\mathrm{crit}_{\tilde{S}}(\varphi)$ are hyperbolic equilibria of (23) and (23) converges to $\mathrm{crit}_{\tilde{S}}(\varphi)$. Moreover, the equilibria $\tilde{Z} \in \mathrm{crit}_{\tilde{S}}(\varphi) \setminus \{R^*\}$ of (23) are unstable whereas the equilibrium $\tilde{Z} = R^*$ of (23) is asymptotically stable.

Remark 1. Note that the extension of this theory to SO(3) is nontrivial as Koditschek and Rimon [1990] cover only differential equations with Euclidean state-spaces. The restriction of the flow of the gradient system for the navigation function does not necessarily inherit the convergence properties from the ambient space. Therefore, Lemma 2 does not hold true directly for the restriction of $-\nabla\varphi$ to SO(3). We want to briefly illustrate this on an example; Therefore, let $R^* = I$ and, for simplicity, $m = 0$. This is the simplest case of the considered setup, where the target configuration lies in the center of the workspace and there is no obstacles obstructing us from getting there. Yet, even in this case, we have $\frac{1}{2}\nabla\varphi(R) = -\left(d_I^k(R) - d_I(R) + r_0^2\right)^{1/k} I - d_I(R)\frac{1}{k}\left(d_I^k(R) - d_I(R) + r_0^2\right)^{1/k-1}\left(-kd_I^{k-1}(R) + 1\right) I$. With $\mathrm{P}_R X = \frac{1}{2}R\left(R^\top X - X^\top R\right)$, this is $\operatorname{grad}\varphi(R) = \mathrm{P}_R\nabla\varphi(R) = \left(d_I^k(R) - d_I(R) + r_0^2\right)^{1/k}\left(R^2 - I\right) + d_I(R)\frac{1}{k}\left(d_I^k(R) - d_I(R) + r_0^2\right)^{1/k-1}\left(-kd_I^{k-1}(R) + 1\right)\left(R^2 - I\right)$. Thus, $\operatorname{grad}\varphi(R)$ vanishes whenever $R^2 = I$, i.e. when R is symmetric. Apart from the desired configuration $R = I$, symmetry of R is also satisfied on a connected, compact set. Note that the solution $R = I$ is isolated from this other set. For details on this, we refer to Schmidt et al. [2013a,b]. As, in this particular case, the solution $R = I$ is isolated, we can however infer that we have a unique solution if we choose r_0 sufficiently small, i.e. if we exclude the set of symmetric matrices that not equal the identity from our workspace.

3.2 Practical Stability

Definition 2. A point R^* is said to be practically uniformly stable with respect to (19) under (20), if for every $\varepsilon \in (0,\infty)$, there exist $\delta \in (0,\infty)$ and $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_0,\infty)$, $\tilde{R}_0 \in \tilde{B}_{R^*}^{\delta}$ implies $\tilde{R}(t) \in \tilde{B}_{R^*}^{\varepsilon}$. Equivalently, a point R^* is said to be practically uniformly stable with respect to (4) under (5), if for every $\varepsilon \in (0,\infty)$, there exist $\delta \in (0,\infty)$ and $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_0,\infty)$, $R_0 \in B_{R^*}^{\delta}$ implies $R(t) \in B_{R^*}^{\varepsilon}$.

Definition 3. A point R^* is said to be practically uniformly attractive with respect to (19) under (20), if there exists a $\delta \in (0,\infty)$, such that for every $\varepsilon \in (0,\infty)$, there exist $t_f \in [0,\infty)$ and $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_0,\infty)$, $t \in [t_0 + t_f, \infty)$, $\tilde{R}_0 \in \tilde{B}_{R^*}^{\delta}$ implies $\tilde{R}(t) \in \tilde{B}_{R^*}^{\varepsilon}$. Equivalently, a point R^* is said to be practically uniformly attractive with respect to (4) under (5), if there exists a $\delta \in (0,\infty)$, such that for every $\varepsilon \in (0,\infty)$, there exist $t_f \in [0,\infty)$ and $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_0,\infty)$, $t \in [t_0 + t_f, \infty)$, $R_0 \in B_{R^*}^{\delta}$ implies $R(t) \in B_{R^*}^{\varepsilon}$.

Definition 4. A point R^* is said to be practically uniformly attractive on M with respect to (4) under (5), if, for every compact subset $K \subset M$, for every $\varepsilon \in (0,\infty)$, there exist $t_f \in [0,\infty)$ and $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_0,\infty)$, $t \in [t_0 + t_f, \infty)$, $R_0 \in K$ implies $R(t) \in B_{R^*}^{\varepsilon}$.

Remark 2. Note that Definition 4 implies Definition 3 if there exists $\delta \in (0,\infty)$ such that M contains the closure of $B_{R^*}^{\delta}$. This is because we can choose K in Definition 4 to be the closure of $B_{R^*}^{\delta}$. Vice versa, Definition 3 implies Definition 4 if we set $M = B_{R^*}^{\delta}$.

Definition 5. A point is said to be practically uniformly asymptotically stable with respect to (19) under (20), if it is practically uniformly stable with respect to (19) under (20) and practically uniformly attractive with respect to (19) under (20). Equivalently, a point is said to be practically uniformly asymptotically stable with respect to (4) under (5), if it is practically uniformly stable with respect to (4) under (5) and practically uniformly attractive with respect to (4) under (5).

For the definitions above, we omit the term “practically”, if the right-hand side of the differential equation under consideration does not depend on ω.

3.3 Extremum Seeking

For the system (23), one needs to feed back $\nabla\varphi(R)$. However, we are interested in scalar feedbacks (20). To construct such feedbacks, Dürr et al. [2013b] proposed the so-called extremum seeking feedback

$$\tilde{U}\left(\omega,\tilde{\xi}\left(\tilde{R}\right),t\right)=\sum_{i=1}^{3}\sum_{j=1}^{3}E_{ij}\left(\varphi\left(\tilde{R}\right)\sqrt{\omega_{ij}}\cos\left(\omega_{ij}t\right)+\sqrt{\omega_{ij}}\sin\left(\omega_{ij}t\right)\right)\qquad(24)$$

to approximate (23), where the matrices E_{ij} form an orthogonal basis of $\mathbb{R}^{3\times 3}$. We can set $\tilde{\xi}=\varphi$ and we assume that all ω_{ij} are chosen to be nonidentical and rational multiples of ω. The following results were proven.

Lemma 3. (Dürr et al. [2013b]). Consider (23) and (19) under feedback (24). Let there exist $B\subset\mathbb{R}^{3\times 3}$ such that $\tilde{Z}(t_0)\in B$ implies $\tilde{Z}(t)\in\tilde{B}_0^A$ with $A\in[0,\infty)$. Then for every bounded $K\subset B$, $D\in(0,\infty)$, $t_f\in(0,\infty)$, there exists an $\omega_0\in(0,\infty)$ such that for every $\omega\in(\omega_0,\infty)$, $t_0\in\mathbb{R}$, $\tilde{Z}_0=\tilde{R}_0\in K$, $t\in\left[t_0,t_0+t_f\right]$, $d_{\tilde{Z}(t)}\left(\tilde{R}(t)\right)\in[0,D)$.

Lemma 4. (Dürr et al. [2013b]). If a point is asymptotically stable for (23), then it is practically uniformly asymptotically with respect to (19) under (24).

Corollary 5. (Dürr et al. [2013a]). The point $\tilde{R}=R^*$ is practically uniformly asymptotically stable with respect to (19) under (24).

By these results, the problem of constructing a feedback (20) such that R^* is a practically uniformly asymptotically stable equilibrium of system (19) (i.e. in the ambient space) is solved. The problem of constructing a feedback (5) such that R^* is a practically uniformly asymptotically stable equilibrium of system (4) (i.e. on SO(3)) remains open.

4. MAIN RESULT

Dürr et al. [2013b] proposed (24) as a scalar feedback to make R^* a practically uniformly asymptotically stable equilibrium of (19). We will propose a scalar feedback (5) to make R^* a practically uniformly asymptotically stable equilibrium of (4). In particular, we propose the extremum seeking feedback $U:\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}\to\mathfrak{so}(3)$

$$U(\omega,\xi(R),t)=\sum_{i=1}^{3}\Omega_i\left(\varphi(R)\sqrt{\omega_i}\cos(\omega_i t)+\sqrt{\omega_i}\sin(\omega_i t)\right)\qquad(25)$$

to approximate the gradient system

$$\dot{Z}=-\operatorname{grad}\varphi(Z),\ \ Z(t_0)=Z_0.\qquad(26)$$

We can set $\xi=\varphi$ and we have to assume that all ω_i are chosen to be nonidentical and rational multiples of ω.

Theorem 6. Consider (26) and (4) under feedback (25). For every $K\subset$SO(3), $D\in(0,\infty)$, $t_f\in(0,\infty)$, there exists an $\omega_0\in(0,\infty)$ such that for every $\omega\in(\omega_0,\infty)$, $t_0\in\mathbb{R}$, $Z_0=R_0\in K$, $t\in\left[t_0,t_0+t_f\right]$, $d_{Z(t)}(R(t))\in[0,D)$.

Proof. Consider

$$\dot{Y}=\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right],\ \ Y(t_0)=Y_0\qquad(27)$$

and (4) under feedback (25), where $[\cdot,\cdot]$ is the Lie bracket of vector fields. As a consequence of Lemma 3, for every $K\subset$SO(3), $D\in(0,\infty)$, $t_f\in(0,\infty)$, there exists an $\omega_0\in(0,\infty)$ such that for every $\omega\in(\omega_0,\infty)$, $t_0\in\mathbb{R}$, $Y_0=R_0\in K$, $t\in\left[t_0,t_0+t_f\right]$, $d_{Y(t)}(R(t))\in[0,D)$. We clearly have

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=\frac{1}{2}\sum_{i=1}^{3}\operatorname{tr}\left(\Omega_iY^\top\nabla\varphi(Y)\right)Y\Omega_i.\qquad(28)$$

We then use the decomposition $\mathbb{R}^{3\times 3}=\mathrm{T}_Y\mathrm{SO}(3)\oplus\mathrm{T}_Y^\perp\mathrm{SO}(3)$ to write

$$\nabla\varphi(Y)=\operatorname{grad}\varphi(Y)+\operatorname{grad}^\perp\varphi(Y)\qquad(29)$$

with $\operatorname{grad}\varphi(Y)\in\mathrm{T}_Y\mathrm{SO}(3)$ and $\operatorname{grad}^\perp\varphi(Y)\in\mathrm{T}_Y^\perp\mathrm{SO}(3)$. We moreover have $Y\Omega_i\in\mathrm{T}_Y\mathrm{SO}(3)$ and hence have the identity $\operatorname{tr}\left(\Omega_iY^\top\operatorname{grad}^\perp\varphi(Y)\right)=0$, which follows from the orthogonality of $\mathrm{T}_Y\mathrm{SO}(3)$ and $\mathrm{T}_Y^\perp\mathrm{SO}(3)$. Hence,

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=\frac{1}{2}\sum_{i=1}^{3}\operatorname{tr}\left(\Omega_iY^\top\operatorname{grad}\varphi(Y)\right)Y\Omega_i.\qquad(30)$$

We know that a tangent vector of SO(3) at a point $Y\in$SO(3) has the form $Y\Omega$ with $\Omega\in\mathfrak{so}(3)$. Therefore, can write $\operatorname{grad}\varphi(Y)=Y\Omega_\varphi$, with $\Omega_\varphi\in\mathfrak{so}(3)$. Thus

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=\frac{1}{2}\sum_{i=1}^{3}\operatorname{tr}\left(\Omega_i\Omega_\varphi\right)Y\Omega_i.\qquad(31)$$

Next, we use a property of the Lie algebra $\mathfrak{so}(3)$. Namely, every element of the algebra can be written as a linear combination of its generators. It is thus possible to write Ω_φ as $\Omega_\varphi=\sum_{j=1}^{3}\Omega_jc_\varphi^j$ to arrive at

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=\frac{1}{2}\sum_{i=1}^{3}\operatorname{tr}\left(\Omega_i\sum_{j=1}^{3}\Omega_jc_\varphi^j\right)Y\Omega_i.\qquad(32)$$

We know that $\operatorname{tr}(\Omega_i\Omega_j)=0$ if $i\neq j$ because two distinct generators are orthogonal to one another and can hence write

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=\frac{1}{2}\sum_{i=1}^{3}\operatorname{tr}\left(\Omega_i^2c_\varphi^i\right)Y\Omega_i.\qquad(33)$$

Moreover, we have $\operatorname{tr}\left(\Omega_i^2\right)=-2$ and can therefore see that

$$\frac{1}{2}\sum_{i=1}^{3}\left[\varphi(Y)Y\Omega_i,Y\Omega_i\right]=-\sum_{i=1}^{3}c_\varphi^iY\Omega_i,\qquad(34)$$

which equals $-\operatorname{grad}\varphi(Y)$. Hence, if $Y_0=Z_0$, then $Y(t)=Z(t)$, which concludes the proof. ■

Theorem 7. The equilibrium $Z=R^*$ of (26) is asymptotically stable. Moreover, every sublevel set of $\varphi(Z)$ is invariant with respect to (26).

Proof. Consider the Lyapunov function candidate

$$V(Z)=\varphi(Z),\qquad(35)$$

which satisfies $V(Z)\geq 0$ and $V(Z)=0$ if and only if $Z=R^*$. We consequently have

$$\dot{V}(Z)=\dot{\varphi}(Z)=\langle\operatorname{grad}\varphi(Z),\dot{Z}\rangle.\qquad(36)$$

Substituting (26), this is

$$\dot{V}(Z)=\dot{\varphi}(Z)=-\langle\operatorname{grad}\varphi(Z),\operatorname{grad}\varphi(Z)\rangle.\qquad(37)$$

Since SO(3) is Riemannian, $\langle\cdot,\cdot\rangle$ is positive definite. Hence, $\dot{V}(Z) \leq 0$ and with (35) it follows that every sublevel set of $\varphi(Z)$ is invariant with respect to (26). In addition, $Z = R^*$ implies $\dot{V}(Z) = 0$. We know from Koditschek and Rimon [1990] that φ is analytic. Moreover, the equations $Z^\top Z - I = 0$ and $\det Z - 1 = 0$ are analytic. Hence $\operatorname{grad}\varphi$ is analytic. Therefore, the solutions to $\operatorname{grad}\varphi(Z) = 0$ can only consist of a finite number of connected components on SO(3) (this is found e.g. in Shiota [1997]). By this argumentation, the solutions to $\dot{V}(Z) = 0$ can only consist of a finite number of connected components on SO(3), as well. By the same argumentation, we know that $\varphi(Z)$ attains a constant value on every connected component of the solution of $\operatorname{grad}\varphi(Z) = 0$ on SO(3). This is a consequence of Morse [1939]. Now note that $\varphi^{-1}(0) = R^*$ is a singleton. Thus, the solution $Z = R^*$ of $\operatorname{grad}\varphi(Z) = 0$ on SO(3) needs to be a singleton as well. Therefore, $\dot{V}$ is negative definite in a neighborhood of R^* and hence the equilibrium $Z = R^*$ of (26) is asymptotically stable. This was the last assertion to be proven. ∎

By the foregoing theorem, every sublevel set of $V(Z) = \varphi(Z)$ is an invariant set of (26). We have $\varphi^{-1}(1) = \partial S$ and can hence conclude that if (26) is initialized in S, then its solution will not enter any O_i at any time.

Theorem 8. The point $R = R^*$ is practically uniformly asymptotically stable with respect to (4) under (25).

Proof. The equilibrium $Z = R^*$ of (26) is asymptotically stable. We have shown that $Y_0 = Z_0$ implies $Y(t) = Z(t)$. From Dürr et al. [2013b], we know that a point which is asymptotically stable for (27) is practically uniformly asymptotically stable with respect to (4) under (25). Last, we know that, by construction, $U : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathfrak{so}(3)$ (because the right-hand side of (25) is a linear combination of the generators of the Lie algebra), such that $\dot{R} \in \mathrm{T}_R\mathrm{SO}(3)$, hence making SO(3) invariant with respect to (4) under (25). ∎

Remark 3. We now know that the solutions of (4) under (25) stay close (in sense of a D-neighborhood) to the solutions of (26) and that, in addition, R^* is practically stabilized. However, (26) may have solutions that come arbitrarily close to the boundary of the obstacles; Therefore, in practice, we can choose choose r_i in a way such that $r_i = r_{i,0} + D$, where $r_{i,0}$ is the physical radius of the obstacle and D is to make the solutions of (4) under (25) not enter the physical obstacle. Yet, another way to approach this issue is addresses in the course of the proof of the next theorem.

It remains to show (9). For doing so, we use a result of Bhatia and Szegő [1967]; Namely, if a point is asymptotically stable with region of attraction $\mathscr{A}$, then it is uniformly attractive on $\mathscr{A}$. Thus, as we have shown in Theorem 7 that the equilibrium $Z = R^*$ of (26) is asymptotically stable, we know that its region of attraction $\mathscr{A}$ is nonempty. Hence, we know moreover, that R^* is uniformly attractive on $\mathscr{A}$ with respect to (26). In the following, let A denote $A = \mathscr{A} \cap \operatorname{int}(S)$. Hence note that $Z = R^*$ is also uniformly attractive on A with respect to (26).

Theorem 9. $R = R^*$ is practically uniformly attractive on A with respect to (4) under (25). Moreover, for every compact subset $K \subset A$, there exists an $\omega_0 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $t \in [t_0,\infty)$, $\omega \in (\omega_0,\infty)$, $R_0 \in K$ implies $R(t) \in \operatorname{int}(S)$.

Proof. Set $\varepsilon_1 = \min(\{d_{R^*}(a)\,|\,a \in \partial S\})$. Existence of ε_1 is ensured by compactness of ∂S. We know from Theorem 8 that $R = R^*$ is practically uniformly asymptotically stable with respect to (4) under (25) and hence know that there exist $\delta_1 \in (0,\infty)$ and $\omega_1 \in (0,\infty)$, such that for all $t_0 \in \mathbb{R}$, $\omega \in (\omega_1,\infty)$, $R_0 \in B_{R^*}^{\delta_1}$ implies $R(t) \in B_{R^*}^{\varepsilon_1}$ for $t \in [t_0,\infty)$. Now choose $\delta_2 \in (0,\delta_1)$ and a compact subset $K \subset A$. For every such K, there exists a $t_1 \in \mathbb{R}$, such that for all $t_0 \in \mathbb{R}$, $t \in [t_0+t_1,\infty)$, $Z_0 \in K$ implies $Z(t) \in B_{R^*}^{\delta_2}$. We now set $D_1 = \delta_1 - \delta_2$, let S_k denote the smallest sublevel set containing K and set

$$D_2 = \min(\{d_a(b)\,|\,a \in \partial S_k, b \in \partial S\}), \tag{38}$$

for our choice of K. D_2 exists due to compactness of S_k and ∂S. Next, set $D_0 = \min(\{D_1, D_2\})$. By means of Theorem 6, there exists $\omega_0 \in (\omega_1,\infty)$ such that for every $\omega \in (\omega_0,\infty)$, $t_0 \in \mathbb{R}$, $R_0 = Z_0 \in K$, $t \in [t_0, t_0+t_1]$, $d_{Z(t)}(R(t)) \in [0,D_0)$. Because $S_k \subset \operatorname{int}(S)$ is invariant with respect to (26), which follows from Theorem 7, and $Z = R^*$ is uniformly attractive on A with respect to (26), we have $R(t) \in \operatorname{int}(S)$ for all $t \in [t_0, t_0+t_1]$ with $\omega \in (\omega_0,\infty)$. For $\omega \in (\omega_0,\infty)$, we hence have $R(t_0+t_1) \in B_{R^*}^{\delta_1}$, which implies $R(t) \in B_{R^*}^{\varepsilon_1} \subset \operatorname{int}(S)$ for $t \in [t_0+t_1,\infty)$. As we have already shown that $R(t) \in \operatorname{int}(S)$ for all $t \in [t_0, t_0+t_1]$, this concludes the proof. ∎

Remark 4. If $S \setminus A$ has measure zero in SO(3), the above implies almost global convergence. If φ is a navigation function on S, then $S \setminus A$ has measure zero; Under additional assumptions, it is possible to show that φ is a navigation function on S. We have moved this result to Appendix A.

5. NUMERICAL EXAMPLE

We want to illustrate our main result on a numerical example. Therefore, we solve the differential equation (4) under (25) with design parameters $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$ and $\omega = 50$ ($\omega_i = \alpha_i\omega$) in MATLAB with `ode15s`. Therein, we choose the initial value $R_0 = I$, and the reference and obstacle configurations

$$R^* = \begin{bmatrix} 0.6428 & 0.6634 & 0.383 \\ -0.766 & 0.5567 & 0.3214 \\ 0 & -0.5 & 0.866 \end{bmatrix}, \; R_1 = \begin{bmatrix} 0.866 & 0.4924 & 0.0868 \\ -0.5 & 0.8529 & 0.1504 \\ 0 & -0.1736 & 0.9848 \end{bmatrix}, \tag{39}$$

respectively, with radius $r_1 = 0.05$. We have purposely chosen R_1 in a way such that it is located "between" R_0 and R^*. All configurations are plotted in Fig. 1 together with the numerical solution of (4) under (25). It can be seen that the solution $R(t)$ performs oscillatory motion and approaches the reference configuration R^*. Moreover, as expected, the solution avoids the obstacle R_1.

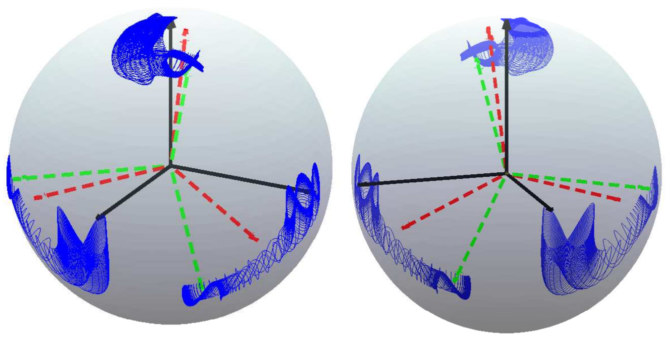

Fig. 1. The solution $R(t)$ (—) of (4) under (25) with initial value R_0 (—) approaches the reference configuration R^* (—) and avoids the obstacle R_1 (—). We have plotted two points of view for better illustration. To plot the rotation matrices, we have multiplied them with the unit vectors of $\mathbb{R}^3$ and depicted the resulting vectors.

6. CONCLUSIONS AND OUTLOOK

We have motivated a scenario where scalar feedbacks shall be used to navigate a point on the rotation matrices to another point whilst avoiding obstacles. Using the principles of navigation functions and extremum seeking systems, we were able to construct such a feedback. We could prove the convergence and stability properties of the resulting closed-loop system. The theoretical results were validated on a suitable numerical example. Future research will rely on proper distance functions on SO(3) rather than on distance functions induced by the ambient space.

Appendix A

Theorem 10. If $\mathrm{crit}_S(\varphi) = \mathrm{crit}_{\tilde{S}}(\varphi)$, then φ is a navigation function on S for sufficiently large k.

Proof. By construction, φ is analytic, polar, and admissible on S. The Hessian of φ in direction $R\Omega$ is given by

$$\mathrm{Hess}\,\varphi(R)(R\Omega) = \mathrm{P}_R\nabla^2\varphi(R)(R\Omega) + \mathfrak{A}_R\left(R\Omega, \mathrm{P}_R^{\perp}\nabla\varphi(R)\right),$$

where $\mathrm{P}_R : \mathbb{R}^{3\times 3} \to \mathrm{T}_R\mathrm{SO}(3)$ is the orthogonal projection from the ambient space to the tangent space at R, $\mathrm{P}_R^{\perp} : \mathbb{R}^{3\times 3} \to \mathrm{T}_R^{\perp}\mathrm{SO}(3)$ is the orthogonal is the orthogonal projection from the ambient space to the normal space at R, and $\mathfrak{A}_R : \mathrm{T}_R\mathrm{SO}(3)\times\mathrm{T}_R^{\perp}\mathrm{SO}(3)\to\mathrm{T}_R\mathrm{SO}(3)$ is the Weingarten map of SO(3) at R. This relation is particularly discussed in Absil et al. [2013]. On SO(3), $\mathrm{P}_R X = \frac{1}{2}R\left(R^\top X - X^\top R\right)$, and hence

$$\mathrm{P}_R\nabla^2\varphi(R)(R\Omega) = \frac{1}{2}\nabla^2\varphi(R)(R\Omega) + \frac{1}{2}R\nabla^2\varphi\left(R^\top\right)(R\Omega)R.$$

Moreover, we have

$$\mathfrak{A}_R(X_1, X_2) = -\frac{1}{2}R\left(X_2^\top X_1 - X_1^\top X_2\right). \tag{A.1}$$

With

$$\mathrm{P}_R^{\perp}\nabla\varphi(R) = \nabla\varphi(R) - \frac{1}{2}\nabla\varphi(R) + \frac{1}{2}R(\nabla\varphi(R))^\top R, \tag{A.2}$$

this is

$$\mathfrak{A}_R\left(R\Omega, \mathrm{P}_R^{\perp}\nabla\varphi(R)\right) = -\frac{1}{4}R(\nabla\varphi(R))^\top R\Omega - \frac{1}{4}\nabla\varphi(R)\Omega$$
$$-\frac{1}{4}R\Omega R^\top\nabla\varphi(R) - \frac{1}{4}R\Omega(\nabla\varphi(R))^\top R.$$

Resubstitution yields $\mathrm{Hess}\,\varphi(R)(R\Omega) = -\frac{1}{4}R(\nabla\varphi(R))^\top R\Omega - \frac{1}{4}R\Omega(\nabla\varphi(R))^\top R + \frac{1}{2}\nabla^2\varphi(R)(R\Omega) + \frac{1}{2}R\nabla^2\varphi\left(R^\top\right)(R\Omega)R - \frac{1}{4}\nabla\varphi(R)\Omega - \frac{1}{4}R\Omega R^\top\nabla\varphi(R)$. We moreover know

$$\mathrm{grad}\,\varphi(R) = \mathrm{P}_R\nabla\varphi(R), \tag{A.3}$$

which can e.g. be found in Absil et al. [2008]. This in turn implies

$$\mathrm{crit}_S(\varphi) = \{R\in S|\nabla\varphi(R) = R(\nabla\varphi(R))^\top R\} \tag{A.4}$$

and we have

$$\mathrm{Hess}\,\varphi(\mathrm{crit}_S(\varphi))(\mathrm{crit}_S(\varphi)\Omega) = \frac{1}{2}\nabla^2\varphi(R)(R\Omega)$$
$$+\frac{1}{2}R\nabla^2\varphi\left(R^\top\right)(R\Omega)R - \frac{1}{2}\nabla\varphi(R)\Omega - \frac{1}{2}R^\top\Omega R\nabla\varphi(R).$$

Hence, by our assumption $\mathrm{crit}_S(\varphi) = \mathrm{crit}_{\tilde{S}}(\varphi)$, we have $\mathrm{Hess}\,\varphi(\mathrm{crit}_S(\varphi))(\mathrm{crit}_S(\varphi)\Omega) = 0$ if and only if

$$\nabla^2\varphi(R)(R\Omega) + R\nabla^2\varphi\left(R^\top\right)(R\Omega)R = 0, \tag{A.5}$$

which is a Sylvester equation with unique solution, because R and $-R$ share no eigenvalues. As

$$\nabla^2\varphi(R)(R\Omega) = 0. \tag{A.6}$$

solves (A.5), we hence know that it is the only solution. It can further be seen that (A.6) holds true if $\Omega = 0$. The only other solution to (A.6) is $\nabla\frac{\partial}{\partial R_{ij}}\varphi(R) = 0$ for all i, j, which contradicts the result of Koditschek and Rimon [1990] that φ is Morse on $\tilde{S}$ for sufficiently large k. Thus, (A.6) has no solution in $\mathrm{crit}_{\tilde{S}}(\varphi)$ and $\Omega = 0$ is the unique solution of (A.5). ■

Remark 5. Naturally, the condition $\mathrm{crit}_S(\varphi) = \mathrm{crit}_{\tilde{S}}(\varphi)$ is restrictive. Yet, we have illustrated in Remark 1 that for a very simple example (i.e. $R^* = I$ and $m = 0$), the condition holds true for sufficiently small r_0.

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