

Extremum Seeking with Drift

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Abstract: We study the convergence properties of extremum seeking controllers when a drift vector field appears in the closed loop. To cope with such issues, we propose a tuning procedure that admits for guaranteeing convergence arbitrarily close to the desired minima despite drift.

Keywords: extremum seeking; drift; practical stability

1. INTRODUCTION

Extremum seeking controllers are feedbacks of the form

$$u_\omega : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad (1)$$

oscillatory in their second argument, parameterized by the frequency $\omega \in (0, \infty)$, that have the purpose of steering the solutions of the system

$$\dot{x} = u(P(x), t) \quad (2)$$

arbitrarily close to the minima of the unknown function $P : \mathbb{R}^n \rightarrow \mathbb{R}$, only with the information provided by the values of P , by choosing sufficiently large ω (we refer the reader to Ariyur and Krstić (2003) for an introduction to the topic). One challenge in extremum seeking is to establish stability properties of (2), such as practical stability (cf. Tan et al. (2006) or Dürr et al. (2013)). The idea in these approaches is to choose u_ω such that solutions of (2) are approximated by solutions of the associated gradient system

$$\dot{y} = -\nabla P(y) =: Y(y). \quad (3)$$

In this paper, we address the question of how to choose u_ω if (2) assumes the form

$$\dot{x} = f(x) + u(P(x), t) =: X(x, t), \quad (4)$$

i.e. if a drift vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ appears in the closed loop, but one still wants to bring solutions of (4) arbitrarily close to the minima of P . As we will see later, this has potential application in problems where f is unknown, for instance if it is subject to a parametric uncertainty (“robust” extremum seeking).

The problem statement resembles the stabilizing extremum seeking problem for input-affine systems posed by Scheinker and Krstić (2013a), who proposed a solution based on control Lyapunov functions and persistence-of-excitation-type conditions on the control vector fields. Yet, in contrast to Scheinker and Krstić (2013a), we will not assume that solutions to (5) approach the minima of P exactly.

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In particular, we study the convergence properties of (4) by introducing the auxiliary gradient system with drift

$$\dot{z} = f(z) - k\nabla P(z) =: Z(z) \quad (5)$$

via first finding sufficiently large k in order to let solutions of (5) approach a neighborhood of the minima of P , such as it was done by Montenbruck et al. (2015), and by then deriving bounds on the proximity of solutions of (4) to solutions of (5) via classical extremum seeking, thus pursuing a two-step procedure. This lets us bring systems with drift arbitrarily close to the minima of a function P only via knowledge of the values of P .

Throughout the manuscript, we assume the twice continuously differentiable potential function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ given such that $M \subset \mathbb{R}^n$ is an asymptotically stable invariant set of (3). Our goal is to consequently find a function (1) such that one can bring the solutions of (4) “close” to M , as $t \rightarrow \infty$, for some given twice continuously differentiable drift vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Notation. By $\nabla P : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we mean the unique vector field which satisfies

$$\nabla P(x) \cdot v = \lim_{h \rightarrow 0} \frac{\nabla P(x + hv) - P(x)}{h} \quad (6)$$

for any x and $v \in \mathbb{R}^n$, where “ \cdot ” denotes the dot product. We denote the solution of (4) initialized at x_0 by $\varphi_x : (x_0, t) \mapsto \varphi_x(x_0, t)$, the solution of (3) initialized at y_0 by $\varphi_y : (y_0, t) \mapsto \varphi_y(y_0, t)$, and the solution of (5) initialized at z_0 by $\varphi_z : (z_0, t) \mapsto \varphi_z(z_0, t)$. For a function such as $P : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote its sublevel sets by $U_P^\alpha = \{x \in \mathbb{R}^n | P(x) \leq \alpha\}$. For a set such as $M \subset \mathbb{R}^n$, we denote its equidistant neighborhood by $U_M^\epsilon = \{x \in \mathbb{R}^n | d(x, M) \leq \epsilon\}$, where, here, d is the infimal Euclidean distance of x to all points in M . Given a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a differentiable function $P : \mathbb{R}^n \rightarrow \mathbb{R}$, the Lie derivative of P along f is $L_f P : \mathbb{R}^n \rightarrow \mathbb{R}$, $x \mapsto \nabla P(x) \cdot f(x)$. Throughout the paper, U will be a neighborhood of M and whenever we write $\text{int } U$, we refer to the interior of U whilst with ∂U , we mean its boundary. For two vector fields f, g , $[f, g]$ denotes the Lie bracket of vector fields. We adopt terminology and results from Bhatia and Szegő (1970). Although we deal with time-dependent vector fields, we omit the dependence on the initial time t_0 in the solutions of the associated differential equations due to the fact that all results hold uniformly in t_0 (cf. Dürr et al. (2013)).

Structure of the Manuscript. We introduce all terminology and auxiliary results that we require in section 2. Thereafter, i.e. in section 3, we present our main results on how to establish convergence guarantees despite drift. We illustrate these results on the example of practical stabilization of the unit sphere despite drift in section 4. Section 5 concludes the manuscript.

2. PRELIMINARIES

The solution that we propose to solve the issue posed in section 1 involves two main ingredients: gradient systems and extremum seeking. We first review some results on gradient systems and consequently repeat the fundamentals of extremum seeking. A key ingredient will be the positive definiteness of P with respect to the desired set M .

Definition 1. A continuously differentiable function $P : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be positive definite with respect to M on U , if $U \subset \mathbb{R}^n$ is a neighborhood of $M \subset \mathbb{R}^n$, P is positive on $U \setminus M$, zero on M , regular on $U \setminus M$, and critical on M .

Having this definition at hand, we repeat a fundamental result about gradient systems, that here only serves the purpose of giving the intuition behind the fact that solutions of (5) approach a neighborhood of M .

Proposition 2. (cf. (Hirsch et al., 2004, Sections 9.2f)). If P is positive definite with respect to M on U and M is compact, then M is an asymptotically stable invariant set of (3) and for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, U_P^α is a subset of the region of attraction of M .

Now, guided by the intuition from perturbation theory (cf. Brauer (1966)), we know that solutions of (5) must stay close to solutions of (3) and thus approach a neighborhood of M whose size can be rendered arbitrarily small by appropriate choice of k . This technique was proposed by Montenbruck et al. (2015). Classically, perturbation theory assumes $f = \epsilon$ constant, whereas, herein, f is allowed to be a vector field.

Lemma 3. If P is positive definite with respect to M on U , M is compact, and f is continuous on U , then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, U_M^ϵ contains an asymptotically stable invariant set of (5) which is also a uniform attractor, and whose region of attraction is a superset of U_P^α .

Proof. The Lie derivative of P along Z is given by $L_Z P(z) = \nabla P(z) \cdot f(z) - k \nabla P(z) \cdot \nabla P(z)$. Choose any $\alpha \in (0, \infty)$ such that U_P^α is compact and $U_P^\alpha \subset U$. As P is positive definite with respect to M on U , for every $\alpha \in (0, \infty)$ such that U_P^α is compact and $U_P^\alpha \subset U$, for any $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists $\delta \in (0, \alpha)$ such that U_P^δ is a subset of U_M^ϵ . It is then true that $U_P^\alpha \setminus \text{int} U_P^\delta$ is a compact, nonempty set. As f is continuous and P is continuously differentiable, $\nabla P \cdot f$ assumes its maximum on $U_P^\alpha \setminus \text{int} U_P^\delta$, which we denote by f_δ^α . It follows that for all $z \in U_P^\alpha \setminus \text{int} U_P^\delta$, $L_Z P(z) \leq f_\delta^\alpha - k \nabla P(z) \cdot \nabla P(z)$. As P is continuously differentiable and positive definite with respect to M on U , for every $\alpha \in (0, \infty)$ such that U_P^α is compact and $U_P^\alpha \subset U$, $\nabla P(z) \cdot \nabla P(z)$ assumes

its positive minimum on $U_P^\alpha \setminus \text{int} U_P^\delta$, which we denote by p_δ^α . It follows that $L_Z P(U_P^\alpha \setminus \text{int} U_P^\delta) \leq f_\delta^\alpha - k p_\delta^\alpha$. Setting $k_0 = f_\delta^\alpha / p_\delta^\alpha$, we have that for any $k \in (k_0, \infty)$, $L_Z P(U_P^\alpha \setminus \text{int} U_P^\delta) < 0$, letting us conclude that U_P^δ is an invariant set of (5). Now define a function as being $P - \delta$ outside U_P^δ and to be zero inside U_P^δ . This function is continuous and its Lie derivative along Z outside U_P^δ equals $L_Z P$. By Lyapunov's direct method, it follows that U_P^δ is an asymptotically stable invariant set of (5). Moreover, as we have $L_Z P(U_P^\alpha \setminus \text{int} U_P^\delta) < 0$, we know that U_P^α is an invariant set. It follows from LaSalle's invariance principle that U_P^α is a subset of the region of attraction of U_P^δ . This concludes the proof. \square

We now repeat some fundamental concepts of extremum seeking, mostly taken from Dürr et al. (2013).

For doing so, define

$$\dot{\xi} = b_0(\xi) + \sum_{j=1}^m b_j(\xi) \sqrt{\omega} v_j(\omega t) \quad (7)$$

and

$$\dot{\zeta} = b_0(\zeta) + \sum_{\substack{j=1 \\ k=j+1}}^m [b_j, b_k](\zeta) \eta_{kj} \quad (8)$$

with

$$\eta_{kj} = \frac{1}{T} \int_0^T v_k(\theta) \int_0^\theta v_j(\tau) d\tau d\theta. \quad (9)$$

Here and henceforth, let $\varphi_\xi : (\xi_0, t) \mapsto \varphi_\xi(\xi_0, t)$ denote the solution of (7) initialized at ξ_0 and $\varphi_\zeta : (\zeta_0, t) \mapsto \varphi_\zeta(\zeta_0, t)$ denote the solution of (8) initialized at ζ_0 . With these auxiliary systems, we repeat two basic results in extremum seeking.

Lemma 4. ((Dürr et al., 2013, Theorem 1)). For all i , let v_i be T -periodic with zero average. For all i , let b_i be twice continuously differentiable. If there exists $B \subset \mathbb{R}^n$ such that there exists $\kappa \in (0, \infty)$ such that for all $\zeta_0 \in B$, for all $t \in [0, \infty)$, $\|\varphi_\zeta(\zeta_0, t)\| < \kappa$, then for every bounded $K \subset B$, for every $D \in (0, \infty)$, for every $t_f \in (0, \infty)$, there exists $\omega_0 \in (0, \infty)$ such that for all $\omega \in (\omega_0, \infty)$, for every $\zeta_0 \in K$, for all $t \in [0, t_f]$, $d(\varphi_\zeta(\zeta_0, t), \varphi_\xi(\zeta_0, t)) < D$.

Definition 5. A set S is said to be an ω -practically uniformly asymptotically stable set of (7), if, for every $\epsilon \in (0, \infty)$, there exists $\delta \in (0, \infty)$ and $\omega_0 \in (0, \infty)$ such that for all $\omega \in (\omega_0, \infty)$, for all $t \in [0, \infty)$, for all $\xi_0 \in U_S^\delta$, $\varphi_\xi(\xi_0, t) \in U_S^\epsilon$, and if there exists $\delta \in (0, \infty)$ such that for every $\epsilon \in (0, \infty)$, there exists $t_f \in [0, \infty)$ and $\omega_0 \in (0, \infty)$ such that for all $\omega \in (\omega_0, \infty)$, for all $t \in [t_f, \infty)$, for all $\xi_0 \in U_S^\delta$, $\varphi_\xi(\xi_0, t) \in U_S^\epsilon$.

Lemma 6. For all i , let v_i be T -periodic with zero average. For all i , let b_i be twice continuously differentiable. If a compact set S is an asymptotically stable invariant set of (8), then it is a ω -practically uniformly asymptotically stable set of (7).

The lemma resembles (Dürr et al., 2013, Theorem 2) and differs from (Dürr et al., 2013, Theorem 2) only by its stability definition. Namely, in contrast to the lemma, which presumes S to be asymptotically stable, (Dürr et al., 2013, Theorem 2) requires S to be an asymptotically stable uniform attractor.

Definition 7. If S is an attractor of (5) and for every $\delta > 0$, for every compact subset K of the region of attraction of S , there exists $t_f \geq 0$ such that for all $t \in (t_f, \infty)$, for all $z_0 \in K$, $\varphi_z(z_0, t) \in U_S^\delta$, then S is called a uniform attractor of (5).

The proof of Lemma 6 thus requires the following lemma.

Lemma 8. ((Bhatia and Szegő, 1970, Theorem V.1.16)). If S is a compact and asymptotically stable invariant set of (5), then S is a uniform attractor of (5).

Proof of Lemma 6. The lemma follows from (Dürr et al., 2013, Theorem 2) after application of Lemma 8. \square

3. MAIN RESULT

We solve the problem from section 1 via a two-step procedure. In particular, we first find sufficiently large k such that solutions of (5) approach the minima of P such as solutions of (3) do. This is done by application of Lemma 3. We second use extremum seeking in order to bring (4) to the form of (7) and to hence keep its solutions in proximity of solutions of (5), which has the form of (8) by finding sufficiently large ω . This is done by application of Lemmata 4 and 6.

In this spirit, we propose a function

$$u_{(k,\omega)} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n, \quad (10)$$

parameterized by $(k, \omega) \in (0, \infty)^2$ to solve the problem from section 1. More particular, let

$$u_{(k,\omega)}(P(x), t) = \sum_{i=1}^n e_i \left(P(x) \sqrt{i\omega} \sin(i\omega t) - 2k\sqrt{i\omega} \cos(i\omega t) \right) \quad (11)$$

with e_1, \dots, e_n being orthonormal vectors of \mathbb{R}^n that satisfy $\text{span}\{e_1 \dots e_n\} = \mathbb{R}^n$. With this choice of u at hand, we are able to state our main results.

Our first result regards reachability of every (arbitrarily small) proximity of M by choosing sufficiently large k, ω .

Theorem 9. If P is positive definite with respect to M on U , M is compact, and f is twice continuously differentiable, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$ and for every $t_f \in (0, \infty)$ there exists an $\omega_0 \in (0, \infty)$ such that for every $\omega \in (\omega_0, \infty)$ and for every $x_0 \in U_P^\alpha$, $\varphi_x(x_0, t_f) \in U_M^\epsilon$.

Proof. For any compact $U_P^\alpha \subset U$, choose some $\epsilon \in (0, d(M, \partial U_P^\alpha))$. Now choose $D < \epsilon$. By virtue of Lemma 3, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, $U_M^{\epsilon-D}$ contains an asymptotically stable invariant set of (5), which we denote by S , whose region of attraction is a superset of U_P^α .

Now, as M is compact, $U_M^{\epsilon-D}$ is compact, and thus, S is compact. Define $\delta = d(S, \partial U_M^{\epsilon-D})$ (such a δ exists by virtue of the aforementioned compactnesses).

As U_P^α is compact, and as it is moreover a superset of the region of attraction of S , it follows from Lemma 8, that there exists t_f' such that for all $t \in (t_f', \infty)$, for all $z_0 \in U_P^\alpha$, $\varphi_z(z_0, t) \in U_S^\delta$. By our very choice of δ , we moreover have that $U_S^\delta \subset U_M^{\epsilon-D}$.

Next, let k be fixed but greater than the above k_0 . We consider (4) under (11) which is

$$\dot{x} = f(x) + \sum_{i=1}^n e_i P(x) \sqrt{i\omega} \sin(i\omega t) - 2ke_i \sqrt{i\omega} \cos(i\omega t).$$

We now see that (3) can be written in the form (7) by setting $m = 2n$ and identifying $b_0 = f$, $b_{2i-1} = e_i P$, $b_{2i} = 2ke_i$, $i = 1, \dots, n$. The corresponding Lie bracket system (8) then coincides with (5) which is due to the fact that the frequencies of the perturbations \sin and \cos are different. Then we have $\varphi_z = \varphi_\zeta$ and $\varphi_x = \varphi_\xi$ with the property that for all i , v_i is T -periodic and has zero average. Now choose any $t_f \in (t_f', \infty)$. As there exists $B \subset \mathbb{R}^n$ such that for all $z_0 \in B$, $\varphi_\zeta(\zeta_0, t) = \varphi_z(z_0, t)$ is uniformly bounded on $[0, \infty)$, namely $B = U_P^\alpha$, application of Lemma 4 yields $\omega_0 \in (0, \infty)$ such that for all $\omega \in (\omega_0, \infty)$, for every $x_0 \in U_P^\alpha$, for all $t \in [0, t_f]$, $d(\varphi_x(x_0, t), \varphi_z(x_0, t)) < D$. As we had shown before that for all $t \in (t_f', \infty)$, for all $z_0 \in U_P^\alpha$, $\varphi_z(z_0, t) \in U_M^{\epsilon-D}$, and as $t_f > t_f'$, this reveals that for all $x_0 \in U_P^\alpha$, $\varphi_x(x_0, t_f) \in U_M^\epsilon$, which was to be proven. \square

Our second result states that every (arbitrarily small) neighborhood of M contains ω -practically uniformly asymptotically stable sets when choosing sufficiently large k .

Theorem 10. If P is positive definite with respect to M on U , M is compact, and f is twice continuously differentiable, then, for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, U_M^ϵ contains an ω -practically uniformly asymptotically stable set of (4).

Proof. Application of Lemma 3 reveals that for every compact $U_P^\alpha \subset U$, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, U_M^ϵ contains an asymptotically stable invariant set of (5), which we denote by S . We now see that (3) can be written in the form (7) by setting $m = 2n$ and identifying $b_0 = f$, $b_{2i-1} = e_i P$, $b_{2i} = 2ke_i$, $i = 1, \dots, n$. The corresponding Lie bracket system (8) then coincides with (5) which is due to the fact that the frequencies of the perturbations \sin and \cos are different. Then we have $\varphi_z = \varphi_\zeta$ and $\varphi_x = \varphi_\xi$ with the property that for all i , v_i is T -periodic and has zero average. By virtue of Lemma 6, S is an ω -practically uniformly asymptotically stable set of (4). This concludes the proof. \square

In both results, we rely on the existence of some positive α such that $U_P^\alpha \subset U$ and U_P^α is compact. The results of Wilson (1967) would shed light on the existence of such α for the case that M be a compact submanifold of \mathbb{R}^n . For the sake of self-containedness, we yet also include such an existence result here for rather general compact M .

Proposition 11. (cf. (Bhatia and Szegő, 1970, Theorem VIII.2.5)). Let $P : U \rightarrow \mathbb{R}$ be continuously differentiable with $U \subset \mathbb{R}^n$ open and let $M \subset U$ be compact. Let P be positive definite with respect to M on U . Then, there exists a $\delta_0 > 0$ such that $\text{int } U_M^{\delta_0} \subset U$ and for all $\delta \in (0, \delta_0)$, there exists an $\alpha > 0$ such that $(U_P^\alpha \cap U_M^{\delta_0}) \subset U_M^\delta$. Moreover, if $U = \mathbb{R}^n$, then $U_P^\alpha \cap U_M^{\delta_0}$ is a compact, isolated component of U_P^α .

Proof. First, choose some $\delta_0 > 0$ such that $U_M^{\delta_0} \subset U$, which exists due to the fact that $M \subset U$, M is compact and U is open.

Second, we show that for all $\delta \in (0, \delta_0)$ there exists an $\alpha > 0$ such that $U_P^\alpha \cap U_M^{\delta_0} \subset U_M^\delta$. Suppose for the sake of contradiction that there exists a $\delta \in (0, \delta_0)$ such that for all $\alpha > 0$ there exists an $x \in U_P^\alpha \cap U_M^{\delta_0}$ such that $x \notin U_M^\delta$.

Then define a sequence $(\alpha_n)_{n \in \mathbb{N}}$ such that $\alpha_n > 0$ and $\alpha_n \rightarrow 0$. For each of the α_n there exists a $x_n \in U_P^{\alpha_n} \cap U_M^{\delta_0}$ such that $x_n \notin U_M^\delta$. Now, since $U_M^{\delta_0}$ is bounded, by the Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $x_{n_k} \rightarrow x_\infty$ for some $x_\infty \in U_M^{\delta_0}$. However, by continuity of P , we have on the one hand that

$$\lim_{k \rightarrow \infty} P(x_{n_k}) = P\left(\lim_{k \rightarrow \infty} x_{n_k}\right) = P(x_\infty) \quad (12)$$

and on the other hand

$$\lim_{k \rightarrow \infty} P(x_{n_k}) \leq \lim_{k \rightarrow \infty} \alpha_{n_k} = 0, \quad (13)$$

thus $x_\infty \in M$. By convergence, there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have that $\|x_{n_k} - x_\infty\| \leq \frac{\delta}{2}$, which leads to the contradiction $x_{n_k} \in U_M^{\delta/2}$, thus proving the claim.

Third, let $U = \mathbb{R}^n$. We observe that since P is continuous, U_P^α is closed for all $\alpha > 0$ (Rudin, 1964, Corollary of Theorem 4.8). Now, since $\delta < \delta_0$ we have that $U_P^\alpha \cap U_M^{\delta_0}$ is bounded and hence compact. In particular, since $\delta < \delta_0$, there exists a $\delta_1 \in (\delta, \delta_0)$ such that

$$U_{(U_P^\alpha \cap U_M^{\delta_0})}^{(\delta_1 - \delta)} \subset \left(U_P^\alpha \cap U_M^{\delta_0}\right), \quad (14)$$

i.e., $U_P^\alpha \cap U_M^{\delta_0}$ is an isolated component of U_P^α . This was the last statement to be proven. \square

Together with this latter proposition, our main results endow one with the ability to choose k and ω appropriately in order to not only let solutions of (4) under (11) reach arbitrarily small neighborhoods of M , but also to remain there in a practically stable fashion. This solves the control problem from section 1.

Our approach has potential application in problems where f is unknown, but parametrized by a bounded parameter, which we term ‘‘robust’’ extremum seeking. In particular, assume that f is subject to a parametric uncertainty, i.e. that f is parameterized via a parameter $\mu \in \mathbb{R}^m$, $f(x) = f(x, \mu)$. If f is continuous in μ and μ is restricted to a compact set $\Delta \subset \mathbb{R}^m$, then it is possible to replace f_δ^α in the proof of Lemma 3 by

$$\max_{\substack{z \in U_P^\alpha \setminus \text{int } U_P^\delta \\ \mu \in \Delta}} \nabla P(z) \cdot f(z, \mu) =: f_{\delta, \Delta}^\alpha. \quad (15)$$

This lets one obtain an overestimate

$$k_0 = \frac{f_{\delta, \Delta}^\alpha}{p_\delta^\alpha} \quad (16)$$

which is valid for any $\mu \in \Delta$.

Let $\varphi_{x, \mu} : (x_0, t) \mapsto \varphi_{x, \mu}(x_0, t)$ denote the solution of

$$\dot{x} = f(x, \mu) + u(P(x), t), \quad (17)$$

initialized at x_0 , for some particular $\mu \in P$.

In this setting, it is possible to recast our main results for the robust extremum seeking problem.

Following the course of Theorem 9, we infer that for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, for every $t_f \in (0, \infty)$, and every $\mu \in \Delta$, there exists an $\omega_0 \in (0, \infty)$ such that for every $\omega \in (\omega_0, \infty)$ and for every $x_0 \in U_P^\alpha$, $\varphi_{x, \mu}(x_0, t_f) \in U_M^\epsilon$.

Following the course of Theorem 10, we further have that for every $\alpha \in (0, \infty)$ such that $U_P^\alpha \subset U$ and U_P^α is compact, for every $\epsilon \in (0, d(M, \partial U_P^\alpha))$, there exists a $k_0 \in (0, \infty)$ such that for every $k \in (k_0, \infty)$, for every $\mu \in \Delta$, U_M^ϵ contains an ω -practically uniformly asymptotically stable set of (17).

4. EXAMPLE: THE UNIT CIRCLE

In this example, we apply our above approach to practical stabilization of the unit sphere

$$S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\} \quad (18)$$

(i.e. $n = 2$ and $M = S^1$) despite drift. Stabilization of the unit sphere is, for instance, relevant in artificial pattern generators. To apply our findings to this problem, we need to define a potential function which is positive definite with respect to S^1 on $\mathbb{R}^2 \setminus \{0\}$, for instance

$$P : x \mapsto -\frac{1}{2} \|x\|^2 + \frac{1}{3} \|x\|^3 + \frac{1}{6}. \quad (19)$$

The function P is plotted in Fig. 1.

In this example, we shall be concerned with the exemplary drift vector field

$$f : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \|x\| - x_2 \\ x_2 \|x\| + x_1 \end{bmatrix} \quad (20)$$

under which S^1 is unstable for $u = 0$ and which will turn out to be particularly suited for illustrating the two-step tuning procedure that we proposed, i.e. that there exists a k_0 such that for any $k \in (k_0, \infty)$ there exists an ω_0 such that for any $\omega \in (\omega_0, \infty)$, solutions approach the desired neighborhood $U_{S^1}^\epsilon$, but that it is not in general true that for any k , there exists an ω_0 such that for any $\omega \in (\omega_0, \infty)$, solutions approach the desired neighborhood $U_{S^1}^\epsilon$ (simply said, the two parameters can not be tuned independently). Please note that this choice of f is not twice continuously differentiable (one would have to exclude the origin to obtain this property). We refer to Scheinker and Krstić (2013b) for the extension of extremum seeking to such vector fields and omit the technical discussion here.

We solved the differential equation (4) under the extremum seeking feedback (11) numerically in MATLAB using `ode45` for different values of k and ω and depict the resulting numerical approximation of φ_x for $x_0 = [2 \ 2]^\top$ in Fig. 2. The simulation reveal that for $k = 2$, increasing ω results in a decrease of ϵ , as expected. For $k = 1$, however, ϵ can not be rendered small by choice of ω . This illustrates that first a sufficiently large k (here it is $k \in (1, \infty)$) must be found before ω can be adjusted in order to decrease ϵ as desired; it is yet not true that the latter tuning of ω is feasible for any choice of k (here e.g. not for $k = 1$).

5. CONCLUSION

We studied convergence properties of extremum seeking controllers which are subject to drift. In order to cope with such issues, we presented a framework in which we could bring the solutions of the controlled system arbitrarily close to the minima of a given potential function despite the drift vector field. Our approach can be applied to robust extremum seeking problems in which the drift vector field is unknown but contained in a compact set, for instance when the drift vector field contains an uncertain parameter. We illustrated our findings on a numerical example in which we practically stabilized the unit sphere.

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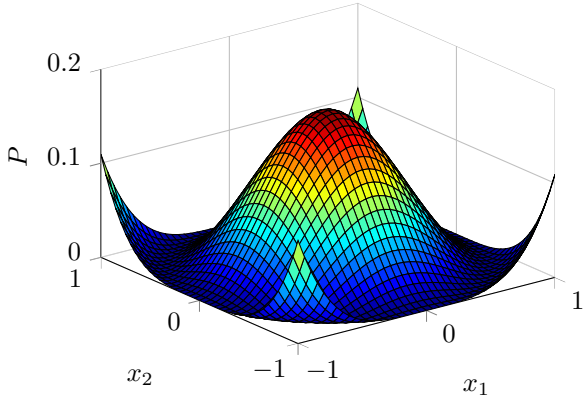


Fig. 1. Plot of the function P as in (19), which is positive definite with respect to S^1 on $\mathbb{R}^2 \setminus \{0\}$

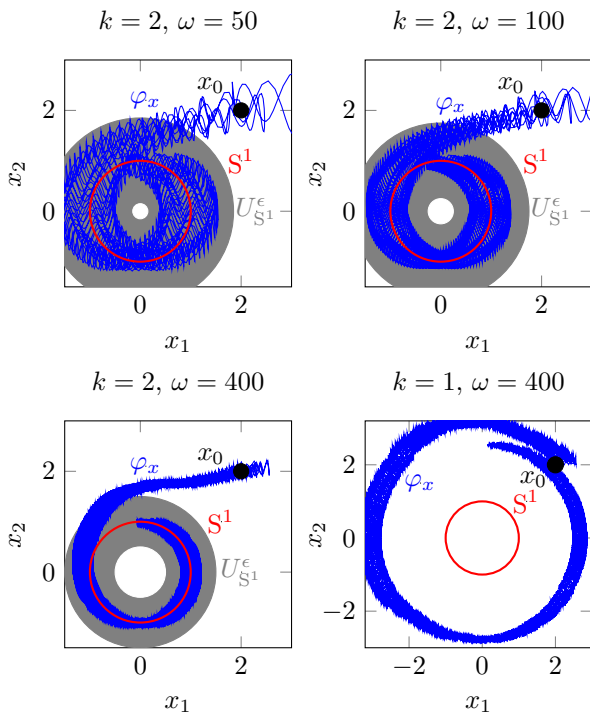


Fig. 2. Numerical approximations of the solutions φ_x to (4) under the extremum seeking feedback (11) for different choices of k and ω for M being the unit sphere S^1