Navigation and Obstacle Avoidance via Backstepping for Mechanical Systems with Drift in the Closed Loop

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***Abstract*— We study backstepping controllers whose goal it is to navigate a mechanical system to a submanifold of its generalized positions under the circumstance that a drift vector field appears in the closed loop due to unmodeled dynamics. We not only derive conditions under which the desired submanifold remains asymptotically stable despite drift, but also provide results on overestimating the navigation function along solutions, such that obstacle avoidance can be guaranteed despite drift.**

I. Introduction

We are concerned with backstepping controllers (cf. [2, Section 6.1]) for mechanical systems in the case when an unknown vector field governs the (generalized) velocities, i.e. when a drift vector field appears in the vector field of the closed loop. More particular, we treat the case where the backstepping controller was designed in order to let the (generalized) positions follow the solutions of a gradient system for some appropriately defined potential function vanishing on a submanifold of the state space. This is desirable in many mechanical systems, e.g. when a navigation or obstacle avoidance problem is to be solved [3]. In this latter context, the potential function is also called a (robot) navigation function.

The problem of unknown velocity vector fields is usually solved using adaptive control laws or nonlinear gains [4]–[7]. In contrast to these works, as we focus on the case where the backstepping controller was designed in order to let the vector field governing the positions take the form of a gradient vector field for some appropriately defined navigation function, we aim to give guarantees on the decrease of the navigation function despite the unknown velocity vector field. This approach bares similarities with the comparison principle [8] (in the sense of overestimating a scalar function by a known function for all times) as well as with the perturbation method [9] (in the sense of exploiting the similarity of solutions of perturbed and unperturbed vector fields). Moreover, we extend previous results from singletons to submanifolds.

In particular, we will present a controller design method where a drift vector field in the velocities may be ignored during controller design, i.e. we draw conclusions from the knowledge of the system without drift to the system with drift.

Meanwhile, all presented results were generalized [1]. The authors are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, and thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart.
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For drift vector fields which are locally Lipschitz continuous, our controller design method is able to attain asymptotic stability of the same invariant submanifold for the system with drift as for the system without drift. Moreover, the proposed method guarantees a decrease of the navigation function of the system with drift in comparison with the navigation function of the system without drift.

Structure of the Paper: We formalize the problem statement in section II. In section III, we review results for backstepping controllers without drift and in section IV, we show that these results extend to practical asymptotic stability for continuous drift. Thereafter, in section V, we prove that it is possible to attain asymptotically stable even in the presence of drift. We give a guarantee on the decrease of the navigation function for the system with drift in section VI and illustrate the proposed method on the example of an obstacle avoidance problem in a terrain with unknown height map in section VII. The paper concludes with section VIII.

Notation: Given a function $P:\mathbb{R}^n\to\mathbb{R}$, $\mathscr{U}_P^\alpha=\{x\in\mathbb{R}^n|P(x)\leq\alpha\}$. When P is differentiable, $\nabla P:\mathbb{R}^n\to\mathbb{R}^n$ denotes the unique vector field satisfying $\nabla P(x)\cdot v=\lim_{h\to 0}\frac{P(x+hv)-P(x)}{h}$ for all v, where $\cdot$ is the inner product. Accordingly, when we write $\nabla^2 P:\mathbb{R}^n\to\mathbb{R}^{n\times n}$, we mean the function which has the result of the application of ∇ to the ith element of ∇P as its ith row. When $\mathscr{M}$ is a subset of $\mathbb{R}^n$ and x is an element of $\mathbb{R}^n$, we mean $\inf_{y\in\mathscr{M}}\|x-y\|$ when we write $d(x,\mathscr{M})$. We denote $\mathscr{U}_\mathscr{M}^\alpha=\{x\in\mathbb{R}^n|d(x,\mathscr{M})\leq\alpha\}$. If we write $P(\mathscr{M})=0$ (>0), we mean that for all $x\in\mathscr{M}$, $P(x)=0$ (>0). For the submanifold $\mathscr{M}$ of $\mathbb{R}^n$, $r:\mathscr{T}\to\mathscr{M}$ denotes the smooth retraction from the tubular neighborhood $\mathscr{T}$ of $\mathscr{M}$ onto $\mathscr{M}$ (all presented results are implicitly restricted to tubular neighborhoods of $\mathscr{M}$). Given a function $f:\mathbb{R}^n\to\mathbb{R}^n$, we denote the Lie derivative of P along f by $\mathrm{L}_f P:\mathbb{R}^n\to\mathbb{R}$, $\mathrm{L}_f P(x)=\nabla P(x)\cdot f(x)$. $\dot{x}=f(x)$ represents a differential equation and we expect that f is defined such that the solution $\phi:\mathbb{R}^n\times(-\epsilon,\epsilon)\to\mathbb{R}^n$ (the function satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\phi(x_0,t)=f(\phi(x_0,t))$, where $x_0=\phi(x_0,0)$ is the initial condition) exists at least on $(-\epsilon,\epsilon)$. For $x\in\mathbb{R}^n$, we write x^2 to denote $x\cdot x$.

II. Problem Statement

We are concerned with backstepping controllers for fully actuated mechanical systems of the form

$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix}=\begin{bmatrix}v\\ f(x,v)+\Delta(x)+F\end{bmatrix}. \tag{1}$$

Therein, x are the (generalized) positions, v are the (generalized) velocities, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the known part of the velocity vector field, $\Delta : \mathbb{R}^n \to \mathbb{R}^n$ is the unknown part of the velocity vector field, and F are the (generalized) forces and torques. In particular, we are concerned with controllers that were designed with the goal to let the x-dynamics take the form of a gradient system for some navigation function $P : \mathbb{R}^n \to \mathbb{R}$. A suitable controller design procedure for this class of problems is backstepping [2, Section 6.1]. Applying backstepping with the aim to let x follow "$\dot{x} = -k_1 \nabla P(x)$", where $k_1 > 0$ is the control gain, lets us define the error $e = \dot{x} + k_1 \nabla P(x)$, which we try to bring to zero. The time-derivative of the error is given by $\dot{e} = \dot{v} + k_1 \nabla^2 P(x) v$ and we aim to let $P(x) + \frac{1}{2} e^2$ decrease along solutions. This is achieved if we set $\dot{e} = -\nabla P(x) - k_1 e$. Solving for $\dot{v}$ yields $\dot{v} = -k_1 v - k_1 \nabla^2 P(x) v - \left(k_1^2 + 1\right) \nabla P(x)$ and equating this with $\dot{v} = f(x, v) + F$ yields

$$F = -f(x, v) - \left(1 + k_1^2\right) \nabla P(x) - k_1 v - k_1 \nabla^2 P(x) v. \quad (2)$$

As Δ is unknown, the closed loop attains the form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta(x) \end{bmatrix} - \begin{bmatrix} v \\ \left(1 + k_1^2\right) \nabla P(x) + k_1 v + k_1 \nabla^2 P(x) v \end{bmatrix}$$

where we refer to $\left[\begin{smallmatrix} 0 \\ \Delta \end{smallmatrix}\right]$ as the drift vector field. The system without drift (i.e. for $\Delta = 0$) takes the form

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \eta \\ -\left(1 + k_2^2\right) \nabla P(\xi) - k_2 \eta - k_2 \nabla^2 P(\xi) \eta \end{bmatrix},$$

where we have replaced k_1 by k_2 in order to distinguish the two control laws. In the analysis of these systems, it is convenient to define an error variable $e = v + k_1 \nabla P(x)$ (or $\zeta = \eta + k_2 \nabla P(\xi)$, respectively) to arrive at the formulation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0 \\ \Delta(x) \end{bmatrix} + \begin{bmatrix} e - k_1 \nabla P(x) \\ -\nabla P(x) - k_1 e \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} \zeta - k_2 \nabla P(\xi) \\ -\nabla P(\xi) - k_2 \zeta \end{bmatrix}. \quad (4)$$

Henceforth, we will refer to (3) as the system with drift and to the right-hand side of (3) as $X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (i.e. $(\dot{x}, \dot{e}) =: X(x, e)$), whereas we refer to (4) as the system without drift (or also the nominal system) and to the right-hand side of (4) as $Z : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ (i.e. $\left(\dot{\xi}, \dot{\zeta}\right) =: Z(\xi, \zeta)$). We let $\phi_x((x_0, e_0), t)$ denote the solution of (3) in x and $\phi_e((x_0, e_0), t)$ the solution of (3) in e with initial condition (x_0, e_0), whereas $\phi_\xi((\xi_0, \zeta_0), t)$ denotes the solution of (4) in ξ and $\phi_\zeta((\xi_0, \zeta_0), t)$ the solution of (4) in ζ with initial condition (ξ_0, ζ_0). Our goal is to conclude stability and convergence properties of (3) from properties of Z.

III. The Nominal Case

In mechanical systems, it is often desirable to let the positions ϕ_x follow the solution of a gradient system for some a priori chosen navigation function $P : \mathbb{R}^n \to \mathbb{R}$, i.e. our goal is to use the backstepping controller to let the closed loop (3) approximate "$\dot{x} = -k_1 \nabla P(x)$". This goal makes particular sense when ϕ_x is ought to converge to a submanifold $\mathscr{M} \subset \mathbb{R}^n$ w.r.t. which P is positive definite.

We will assume $\mathscr{M}$ to be a submanifold throughout the paper, even when we do not state it explicitly (this will be required in order to let the retraction onto $\mathscr{M}$ be smooth).

Definition 1 (Positive Definiteness): For a sufficiently smooth function $P : \mathscr{U} \to \mathbb{R}$, P is said to be positive definite with respect to $\mathscr{M} \subset \mathbb{R}^n$ on $\mathscr{U} \subset \mathbb{R}^n$, if $\mathscr{M}$ is compact and connected, $\mathscr{U}$ is a neighborhood of $\mathscr{M}$, $P(\mathscr{M}) = 0 < P(\mathscr{U} \setminus \mathscr{M})$, and $\nabla P(y) = 0 \Leftrightarrow y \in \mathscr{M}$.

When P is chosen such that it is positive definite with respect to $\mathscr{M}$, the backstepping controller (2) renders the submanifold

$$\mathscr{M} \times \{0\} =: \mathscr{N} \subset (\mathbb{R}^n \times \mathbb{R}^n) \quad (5)$$

an asymptotically stable invariant set for the system without drift (4), making ϕ_x converge to $\mathscr{M}$. In the following, let $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote

$$V(\xi, \zeta) = P(\xi) + \frac{1}{2} \zeta^2. \quad (6)$$

Proposition 1 ((cf. [2])): If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$ and $k_2 > 0$, then $\mathscr{N}$ is an asymptotically stable invariant set of (4). Moreover, $\forall \alpha > 0$ such that $\mathscr{U}_V^\alpha \subset \mathscr{U} \times \mathbb{R}^n$, $\mathscr{U}_V^\alpha$ is a subset of the region of asymptotic stability of $\mathscr{N}$.

Proof: Because $\nabla P(\xi) = 0 \Leftrightarrow \xi \in \mathscr{M}$, $\mathscr{N}$ is an invariant set of (4). Further, by definition, $\mathscr{M}$ is compact. Now consider the function (6). V is continuously differentiable, maps a neighborhood of $\mathscr{N}$ to $\mathbb{R}$, and satisfies $\forall \zeta \neq 0 : V(\mathscr{U} \setminus \mathscr{M}, \zeta) > 0$ as well as $V(\mathscr{M}, 0) = 0$. Next, substituting Z, we have

$$\mathrm{L}_Z V(\xi, \zeta) = -k_2 (\nabla P(\xi))^2 - k_2 e^2. \quad (7)$$

Thus, V satisfies $\mathrm{L}_Z V(\xi, \zeta) = 0 \Leftrightarrow \nabla P(\xi) = 0$, $\zeta = 0$. Because P is positive definite with respect to $\mathscr{M}$, thus, $\forall \zeta \neq 0 : \mathrm{L}_Z V(\mathscr{U} \setminus \mathscr{M}, \zeta) < 0$ and $\mathrm{L}_Z V(\mathscr{M}, 0) = 0$. Consequently, applying Lyapunov's direct method and LaSalle's invariance principle, the lemma is proven. ■

Having this result for the system without drift (4) at hand, it is desirable to derive conclusions about the convergence properties of the system with drift (3).

IV. Practical Asymptotic Stability and Drift

Our goal is to let ϕ_x converge to $\mathscr{M}$, i.e. to let $\mathscr{N}$ be an asymptotically stable invariant submanifold of (3), just as it was the case for (4). However, without further assumptions, we may only achieve practical stability,

Definition 2 (Practical Stability): A non-empty compact set $\mathscr{M}$ is said to be a k-practically asymptotically stable set of $\dot{y} = Y_k(y)$, with $Y_k : \mathbb{R}^n \to \mathbb{R}^n$, where k is a parameter, if for every $\alpha > 0$, there exists a $k > 0$, such that $\mathscr{U}_\mathscr{M}^\alpha$ contains an asymptotically stable set of $\dot{y} = Y_k(y)$.

Having the convergence properties of (4) at hand, we can conclude practical convergence properties of (3). This bares similarities with the perturbation method [9] in the sense that the solutions to the system with drift stay "close" to the solutions to the system without drift.

Proposition 2: If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$ and Δ is continuous, then $\mathscr{N}$ is a k_1-practically asymptotically stable set of (3).

Proof: Consider the function (6). Its Lie derivative along X is given by

$$\mathrm{L}_X V(x,e) = -k_1 (\nabla P(x))^2 - k_1 e^2 + e \cdot \Delta(x). \quad (8)$$

Now, introduce $\theta \in (0,1)$ to rewrite the latter as

$$\begin{aligned} \mathrm{L}_X V(x,v) = &- k_1 (1-\theta)(\nabla P(x))^2 - k_1 (1-\theta) e^2 \\ &- k_1 \theta (\nabla P(x))^2 - k_1 \theta e^2 + e \cdot \Delta(x). \end{aligned}$$

Let us define the function $G(x,e)$ as

$$G(x,e) = -k_1 \theta (\nabla P(x))^2 - k_1 \theta e^2 + e \cdot \Delta(x) \quad (9)$$

and see that for $e \neq 0$, $G(x,e) \leq 0$ implies $\mathrm{L}_X V(\mathscr{U} \setminus \mathscr{M}, e) < 0$ and $\mathrm{L}_X V(\mathscr{M}, 0) = 0$ by application of Proposition 1, as P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$. By the inequality of Cauchy-Schwarz,

$$\|\Delta(x)\| \, \|e\| \leq \theta k_1 \big((\nabla P(x))^2 + e^2 \big) \quad (10)$$

implies $G(x,e) \leq 0$. Now note that, because P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, as long as $(x,e) \notin \mathscr{N}$, for every $(x,e) \in \mathscr{U} \times \mathbb{R}^n$, we can find a k_1 such that (10) holds true. More, the dependency of this k_1 on (x,e) is continuous, since Δ is continuous and P is continuously differentiable. Thus, for every compact set C such that $C \cap \mathscr{N} = \emptyset$ and $C \subset \mathscr{U} \times \mathbb{R}^n$, for every $(x,e) \in C$, we may find an overestimate for the left-hand side of (10) and an underestimate for the right-hand side of (10), such that there exists a k_1 to let (10) hold true. Now choose $\alpha > 0$. As P is positive definite with respect to $\mathscr{M}$, it is possible to find a β such that $\mathscr{U}_V^\beta \subset \mathscr{U}_\mathscr{N}^\alpha$. Next, choose $\gamma \geq \beta$ such that $\mathscr{U}_\mathscr{N}^\alpha \subset \mathscr{U}_V^\gamma$. With this at hand, $C = \mathscr{U}_V^\gamma \setminus \mathscr{U}_V^\beta$ is a compact set such that $C \cap \mathscr{N} = \emptyset$. Therefore, it is possible to find k_1 such that for all $(x,e) \in C$, (10) holds true. By the invariance of sublevel sets, $\mathscr{U}_V^\gamma$ and $\mathscr{U}_V^\beta$ are both invariant. By LaSalle's Invariance Principle, this lets $\mathscr{U}_V^\beta$ be asymptotically stable. Moreover, $\mathscr{U}_V^\beta \subset \mathscr{U}_\mathscr{N}^\alpha$, which was to be proven. ■

It is natural to ask for sufficient conditions to let the region of (practical) asymptotic stability from the forgoing proposition extend arbitrarily (semi-globally). One such sufficient condition is radial unboundedness of P (with respect to $\mathscr{M}$).

Definition 3 (Semi-Global Practical Asymptotic Stability): A non-empty compact set $\mathscr{M} \subset \mathbb{R}^n$ is said to be a $\mathscr{U}$-semi-globally k-practically asymptotically stable set of $\dot{y} = Y_k(y)$, where k is a parameter, if for every $\alpha > 0$, for every $\beta > \alpha$ such that $\mathscr{U}_\mathscr{M}^\beta \subset \mathscr{U}$ there exists a $k > 0$, such that $\mathscr{U}_\mathscr{M}^\alpha$ contains an asymptotically stable set of $\dot{y} = Y(y)$ and s.t. $\mathscr{U}_\mathscr{M}^\beta$ is a subset of its region of asymptotic stability.

Definition 4 (Radial Unboundedness): P is said to be $\mathscr{M}$-radially unbounded on $\mathscr{U} \subset \mathbb{R}^n$, if for every $\epsilon > 0$ such that $\mathscr{U}_\mathscr{M}^\epsilon \subset \mathscr{U}$, there exists a $\delta > 0$, such that $\mathscr{U}_\mathscr{M}^\epsilon \subset \mathscr{U}_P^\delta$.

Under the assumptions of the forgoing proposition, with the additional assumption of radial unboundedness of P, it is possible to render $\mathscr{N}$ a semi-globally practically asymptotically stable invariant set.

Corollary 1: If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, P is $\mathscr{M}$-radially unbounded on $\mathscr{U}$, and Δ is continuous, then, for every γ such that $\mathscr{U}_P^\gamma \subset \mathscr{U}$, $\mathscr{N}$ is a $\mathscr{U}_P^\gamma \times \mathbb{R}^n$-semi-globally k_1-practically asymptotically stable set of (3).

Proof: The proof is along the lines of the proof of Proposition 2. Thus, reconsider (10). Again, because P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, as long as $(x,e) \notin \mathscr{N}$, for every $(x,e) \in \mathscr{U} \times \mathbb{R}^n$, we can find a k_1 such that (10) holds true and the dependency of this k_1 on (x,e) is continuous, since Δ is continuous and P is continuously differentiable. Thus, for every compact set C such that $C \cap \mathscr{N} = \emptyset$ and $C \subset \mathscr{U} \times \mathbb{R}^n$, for every $(x,e) \in C$, we may find an overestimate for the left-hand side of (10) and an underestimate for the right-hand side of (10), such that there exists a k_1 to let (10) hold true. To construct such a set C, choose $\alpha > 0$ and $\beta > \alpha$. As P is positive definite with respect to $\mathscr{M}$, it is possible to find a γ such that $\mathscr{U}_V^\gamma \subset \mathscr{U}_\mathscr{N}^\alpha$. As P is $\mathscr{M}$-radially unbounded, we may find a δ such that $\mathscr{U}_\mathscr{N}^\beta \subset \mathscr{U}_V^\delta$. With these values at hand, $C = \mathscr{U}_V^\delta \setminus \mathscr{U}_V^\gamma$ is a compact set such that $C \cap \mathscr{N} = \emptyset$. Therefore, it is possible to find a k_1 such that for all $(x,e) \in C$, (10) holds true. By the invariance of sublevel sets, the sets $\mathscr{U}_V^\gamma$ and $\mathscr{U}_V^\delta$ are both invariant. By Lyapunov's direct method and LaSalle's invariance principle, this lets $\mathscr{U}_V^\gamma$ be asymptotically stable, with $\mathscr{U}_V^\delta$ being a subset of its region of asymptotic stability. Moreover, $\mathscr{U}_V^\gamma \subset \mathscr{U}_\mathscr{N}^\alpha$ and $\mathscr{U}_\mathscr{N}^\beta \subset \mathscr{U}_V^\delta$, which was to be proven. ■

V. Asymptotic Stability and Drift

In the previous section, we have shown that it is possible to conclude practical asymptotic stability of $\mathscr{N}$ of (3) from the assumptions that were sufficient for asymptotic stability of $\mathscr{N}$ of (3) (plus continuity of Δ). It appears natural to ask for sufficient conditions on Δ to conclude asymptotic stability of $\mathscr{N}$ for (3). For doing so, we employ additional smoothness assumptions on Δ.

A. One-Sided Lipschitz Continuous Drift

First, we assume Δ to suffice the one-sided Lipschitz continuity property to later show that this can be relaxed to only requiring Δ to be locally Lipschitz continuous.

Definition 5 (One-Sided Lipschitz Continuity): f is said to be (q)-one-sided Lipschitz continuous on $\mathscr{U}$, if $\exists q : \forall a,b \in \mathscr{U} : (a-b) \cdot (f(a) - f(b)) \leq q (a-b)^2$.

Further, we will require strong convexity of P (with respect to $\mathscr{M}$). When P is positive definite with respect to $\mathscr{M}$, this assumption guarantees (roughly speaking) that $-\nabla P$ never points away from $\mathscr{M}$, i.e. that the angle enclosed between $\nabla P(x)$ and $x - r(x)$, where r denotes the retraction from tubular neighborhoods onto $\mathscr{M}$ (defined via the normal bundle), is acute (when $\mathscr{M}$ is a submanifold, existence of tubular neighborhoods is ensured by the tubular neighborhood theorem).

Definition 6 (Strong Convexity): P is said to be λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}$, if $\mathscr{U}$ is a neighborhood of $\mathscr{M}$ and there exists a $\lambda > 0$, such that for all $x \in \mathscr{U}$, $(\nabla P(r(x)) - \nabla P(x)) \cdot (r(x) - x) \geq \lambda (x - r(x))^2$.

Lemma 1: Let $\mathscr{N}$ be an invariant submanifold of (3). If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, P is λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}$, and $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ is (q)-one-sided Lipschitz continuous on $\mathscr{U}$, then, for $k_1 > \max\{q, \left(\frac{q+1}{2\lambda}\right)^2\}$, $\mathscr{N}$ is an asymptotically stable invariant set of (3) and $\forall \alpha > 0$ such that $\mathscr{U}_V^\alpha \subset \mathscr{U} \times \mathbb{R}^n$, $\mathscr{U}_V^\alpha$ is a subset of the region of asymptotic stability of $\mathscr{N}$.

Proof: Consider the function (6). Its Lie derivative along X is given by

$$\mathrm{L}_X V(x,e) = -k_1 (\nabla P(x))^2 - k_1 e^2 + e \cdot \Delta(x). \quad (11)$$

From $\mathscr{N}$ being an invariant set of (3), as $\nabla P(x) = 0 \Leftrightarrow x \in \mathscr{M}$, it follows that $\Delta(r(x)) = 0$, where r denotes the smooth retraction onto $\mathscr{M}$. Thus, we may write

$$\begin{aligned} e \cdot \Delta(x) &= e \cdot (\Delta(x) - \Delta(r(x))) \quad (12)\\ &= \begin{bmatrix} x - r(x) \\ e - k_1 \nabla P(x) + k_1 \nabla P(x) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \Delta(x) - \Delta(r(x)) \end{bmatrix}. \end{aligned}$$

As, by assumption, $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ is (q)-one-sided Lipschitz continuous on $\mathscr{U}$, we may thus overestimate the latter equation by

$$e \cdot \Delta(x) \leq q (x - r(x))^2 + q e^2. \quad (13)$$

Substituting this into our expression for $\mathrm{L}_X V(x,e)$, we arrive at

$$\mathrm{L}_X V(x,e) \leq -k_1 (\nabla P(x))^2 - k_1 e^2 + q (x - r(x))^2 + q e^2.$$

We further have that

$$k_1 (\nabla P(x))^2 + 2\sqrt{k_1} \nabla P(x) \cdot (x - r(x)) + (x - r(x))^2 \geq 0$$

and hence also

$$\begin{aligned} -k_1 (\nabla P(x))^2 &\leq 2\sqrt{k_1} \nabla P(x) \cdot (x - r(x)) + (x - r(x))^2 \\ &= -2\sqrt{k_1} \nabla P(x) \cdot (r(x) - x) + (x - r(x))^2. \end{aligned}$$

Resubstitution yields

$$\begin{aligned} \mathrm{L}_X V(x,e) \leq &- k_1 e^2 - 2\sqrt{k_1} \nabla P(x) \cdot (r(x) - x) \\ &+ q (x - r(x))^2 + q e^2 + (x - r(x))^2. \end{aligned} \quad (14)$$

By assumption, P is λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}$, such that we have

$$\mathrm{L}_X V(x,e) \leq (q - k_1) e^2 + \left(q - 2\lambda\sqrt{k_1} + 1\right)(x - r(x))^2 \quad (15)$$

after reordering. Choosing $k_1 > \max\{q, \left(\frac{q+1}{2\lambda}\right)^2\}$ yields $\mathrm{L}_X V(x,e) \leq 0$. As $(x - r(x))^2$ is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, we moreover have $\forall e \neq 0$: $\mathrm{L}_X V(\mathscr{U} \setminus \mathscr{M}, e) < 0$ and $\mathrm{L}_X V(\mathscr{M}, 0) = 0$. Applying Lyapunov's direct method and LaSalle's invariance principle proves the claim. ■

B. Locally Lipschitz Continuous Drift

In the foregoing lemma, we have assumed the drift vector field $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ to satisfy the one-sided Lipschitz continuous property, yielding a precise lower bound for k_1. A weaker assumption is the notion of local Lipschitz continuity.

Definition 7 (Local Lipschitz Continuity): f is said to be locally Lipschitz continuous on $\mathscr{U} \subset \mathbb{R}^n$, if for every $x \in \mathscr{U}$, for all y in a neighborhood $\mathscr{U}_x$ of x, there exists a L_x, such that $(f(x) - f(y))^2 \leq L_x (x - y)^2$.

Lemma 2: If f is locally Lipschitz continuous on $\mathscr{U}$, then f is (q)-one-sided Lipschitz continuous on every compact subset of $\mathscr{U}$.

Proof: Choose a compact set $C \subset \mathscr{U}$. Then $\cup_{x \in C} \mathscr{U}_x$ is an open cover of C, i.e. $C \subset \cup_{x \in C} \mathscr{U}_x$. As C is compact, the open cover $\cup_{x \in C} \mathscr{U}_x$ has a finite subcover $\cup_{x \in S} \mathscr{U}_x$, i.e. there exists a finite $S \subset C$ such that $C \subset \cup_{x \in S} \mathscr{U}_x$. It is thus possible to find a value L sufficing

$$L = \max_{x \in S} L_x \quad (16)$$

such that for all x, $y \in C$,

$$(f(x) - f(y))^2 \leq L (x - y)^2. \quad (17)$$

Next, note that

$$(f(x) - f(y))^2 - 2(f(x) - f(y)) \cdot (x - y) + (x - y)^2 \geq 0.$$

This yields

$$2(f(x) - f(y)) \cdot (x - y) \leq (f(x) - f(y))^2 + (x - y)^2,$$

and, together with the Lipschitz inequality, we have

$$(f(x) - f(y)) \cdot (x - y) \leq \frac{L+1}{2} (x - y)^2, \quad (18)$$

which is just the characterization of f being $\left(\frac{L+1}{2}\right)$-one-sided Lipschitz continuous. ■

Lemma 3: If f is locally Lipschitz continuous on $\mathscr{U}$, then $\left[\begin{smallmatrix}0\\ f\end{smallmatrix}\right]$ is locally Lipschitz continuous on $\mathscr{U}$.

Proof: As we have

$$\left(\begin{bmatrix} 0 \\ f(x) \end{bmatrix} - \begin{bmatrix} 0 \\ f(y) \end{bmatrix}\right)^2 = (f(x) - f(y))^2 \quad (19)$$

and for every $x \in \mathscr{U}$, for all y in a neighborhood $\mathscr{U}_x$ of x, there exists a L_x, such that f satisfies

$$(f(x) - f(y))^2 \leq L_x (x - y)^2, \quad (20)$$

we find that for every $x \in \mathscr{U}$, for all y in a neighborhood $\mathscr{U}_x$ of x, there exists a L_x, such that f satisfies

$$\left(\begin{bmatrix} 0 \\ f(x) \end{bmatrix} - \begin{bmatrix} 0 \\ f(y) \end{bmatrix}\right)^2 \leq L_x (x - y)^2. \quad (21)$$

This is just the characterization of $\left[\begin{smallmatrix}0\\ f\end{smallmatrix}\right]$ being locally Lipschitz continuous on $\mathscr{U}$. ■

With the two foregoing lemmata, it is possible to relax the assumption of $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ being one-sided Lipschitz continuous from Lemma 1 to only requiring Δ to be locally Lipschitz continuous.

Theorem 1: Let $\mathscr{N}$ be an invariant set of (3). If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, P is λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}$, and Δ is locally Lipschitz continuous on $\mathscr{U} \times \mathbb{R}^n$, then there exists a k_0, s.t. for all $k_1 > k_0$, $\mathscr{N}$ is an asymptotically stable invariant set of (3).

Proof: The proof is along the lines of the proof of Lemma 1. In particular, we may pursue the prove in the identical fashion up to (12). Now apply Lemma 3 to find that $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ is locally Lipschitz continuous on $\mathscr{U} \times \mathbb{R}^n$. By Lemma 2, $\left[\begin{smallmatrix}0\\ \Delta\end{smallmatrix}\right]$ is (q)-one-sided Lipschitz continuous on every compact subset C of $\mathscr{U} \times \mathbb{R}^n$. We choose a particular such subset to be $C = \mathscr{U}_V^{\alpha}$. In doing so, for all $(x, e) \in C$, it is possible to get back to the proof of Lemma 1 until we arrive at (15). Choosing $k_1 > \max\{q, \left(\frac{q+1}{2\lambda}\right)\}$ yields $\mathrm{L}_X V(C) \leq 0$. As $(x - r(x))^2$ is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, we moreover have $\mathrm{L}_X V(C \setminus \mathscr{N}) < 0$ and $\mathrm{L}_X V(\mathscr{M}, 0) = 0$. As P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, C is a neighborhood of $\mathscr{M} \times \{0\}$. Applying Lyapunov's direct method proves the claim. ∎

Just as in section IV, it is natural to ask for sufficient conditions to let the region of asymptotic stability from the foregoing theorem extend arbitrarily (semi-globally). Again, radial unboundedness of P turns out to be such a condition.

Definition 8 (Semi-Global Asymptotic Stability): Let $\phi_y(y_0, t)$ solve $\dot{y} = Y_k(y)$, where k is a parameter. A non-empty compact set $\mathscr{M}$ is said to be a $\mathscr{U}$-semi-globally asymptotically stable set of $\dot{y} = Y_k(y)$, if for every $\beta > 0$ such that $\mathscr{U}_{\mathscr{M}}^{\beta} \subset \mathscr{U}$ there exists a $k > 0$, such that $\mathscr{M}$ is an asymptotically stable set of $\dot{y} = Y(y)$ and such that $\mathscr{U}_{\mathscr{M}}^{\beta}$ is a subset of the region of asymptotic stability of $\mathscr{M}$.

Theorem 2: Let $\mathscr{N}$ be an invariant submanifold of (3). If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}$, P is λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}$, P is $\mathscr{M}$-radially unbounded on $\mathscr{U}$, and Δ is locally Lipschitz continuous on $\mathscr{U} \times \mathbb{R}^n$, then, for every γ such that $\mathscr{U}_P^{\gamma} \subset \mathscr{U}$, $\mathscr{N}$ is a $\mathscr{U}_P^{\gamma} \times \mathbb{R}^n$-semi-globally asymptotically stable set of (3).

Proof: The proof is along the lines of the proof of Theorem 1. Precisely, choose $\beta > 0$ such that $\mathscr{U}_{\mathscr{M}}^{\beta} \subset \mathscr{U}$. As P is $\mathscr{M}$-radially unbounded on $\mathscr{U}$, there exists an γ such that $\mathscr{U}_{\mathscr{M}}^{\beta} \subset \mathscr{U}_P^{\gamma}$. For every γ such that $\mathscr{U}_P^{\gamma} \subset \mathscr{U}$, as $\mathscr{U}_V^{\gamma} \subset \mathscr{U}_P^{\gamma} \times \mathbb{R}^n$, P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}_V^{\gamma}$. Apply Lemmata 2 and 1 to find that $\left[\begin{smallmatrix}0\\ f\end{smallmatrix}\right]$ is (q)-one-sided Lipschitz continuous on $C = \mathscr{U}_V^{\gamma}$. Then get back to the proof of Theorem 1. Choosing $k_1 > \max\{q, \left(\frac{q+1}{2\lambda}\right)\}$ yields $\mathrm{L}_X V(C) \leq 0$. As P is positive definite with respect to $\mathscr{M}$ on C, we moreover have $\mathrm{L}_X V(C \setminus \mathscr{N}) < 0$ and $\mathrm{L}_X V(\mathscr{M}, 0) = 0$. Using Lyapunov's direct method and LaSalle's invariance principle, we know that for every $\delta \leq \gamma$, $\mathscr{U}_V^{\delta}$ is a subset of the region of asymptotic stability of $\mathscr{N}$. By our construction, $\mathscr{U}_{\mathscr{N}}^{\beta} \subset \mathscr{U}_V^{\gamma}$, making $\mathscr{U}_{\mathscr{N}}^{\beta}$ a subset of the region of asymptotic stability of $\mathscr{N}$. This was our claim. ∎

VI. Guaranteed Decrease of the Navigation Function

The backstepping controller (2) was designed with the goal to let P decrease along ϕ_x.

Although we have found sufficient conditions on k_1 to let V decrease along (ϕ_x, ϕ_e) in the previous section, it remains to show that it is always possible to find a k_1 such that $P(\phi_x((x_0, e_0), t))$ can be quantified. In particular, as (4) represents the known dynamics, it is desirable to be able to compare $P(\phi_x((x_0, e_0), t))$ to $P(\phi_\xi((\xi_0, \zeta_0), t))$. This question bares similarities to the comparison principle [8] in the sense of overestimating a scalar function by a known function for all times.

Theorem 3: Let $\mathscr{N}$ be an invariant submanifold of (3). If P is positive definite with respect to $\mathscr{M}$ on $\mathscr{U}_1$, P is λ-strongly convex with respect to $\mathscr{M}$ on $\mathscr{U}_1$, and Δ is locally Lipschitz continuous on $\mathscr{U}_1 \times \mathbb{R}^n$, then there exists a neighborhood $\mathscr{U}_2$ of $\mathscr{N}$ such that for all $(x_0, e_0) = (\xi_0, \zeta_0) \in \mathscr{U}_2$, for every $T > 0$, there exists a k_0 such that for all $k_1 \geq k_0$, it holds true that $P(\phi_x((x_0, e_0), T)) \leq P(\phi_\xi((\xi_0, \zeta_0), T))$.

Proof: Using Theorem 1, we know that under the assumptions of the theorem, there exists a k_3 such that for all $k_1 > k_3$, $\mathscr{N}$ is an asymptotically stable invariant set of (3). Let $\mathscr{U}_3$ denote a subset of the region of asymptotic stability of $\mathscr{N}$. For all $(x_0, e_0) \in \mathscr{U}_3$, we have

$$\int_0^T \dot{V}(\phi_x((x_0, e_0), \tau), \phi_e((x_0, e_0), \tau))\, \mathrm{d}\tau = V(\phi_x((x_0, e_0), T), \phi_e((x_0, e_0), T)) - V(x_0, e_0).$$

As before, we have the overestimate (15) for the Lie derivative of V along X on every compact subset $\mathscr{U}_2 \subset \mathscr{U}_3$. Substituting the solution, we arrive at the integral inequality

$$\begin{aligned} &\int_0^T \dot{V}(\phi_x((x_0, e_0), \tau), \phi_e((x_0, e_0), \tau))\, \mathrm{d}\tau \\ &\leq \int_0^T (q - k_1)\, \phi_e((x_0, e_0), \tau)^2 + \left(q - 2\lambda\sqrt{k_1} + 1\right) \\ &\qquad (\phi_x((x_0, e_0), \tau) - r(\phi_x((x_0, e_0), \tau)))^2\, \mathrm{d}\tau. \end{aligned} \tag{22}$$

Now, for every $P(\phi_\xi((\xi_0, \zeta_0), T))$, it is possible to choose k_0 such that

$$\begin{aligned} &\int_0^T (q - k_0)\, \phi_e((x_0, e_0), \tau)^2 + \left(q - 2\lambda\sqrt{k_0} + 1\right) \\ &\qquad (\phi_x((x_0, e_0), \tau) - r(\phi_x((x_0, e_0), \tau)))^2\, \mathrm{d}\tau \\ &+ V(x_0, e_0) - \frac{1}{2}\phi_e((x_0, e_0), T)^2 \leq P(\phi_\xi((\xi_0, \zeta_0), T)). \end{aligned} \tag{23}$$

Hence, for every $k_1 \geq k_0$, for all (x_0, e_0) in some neighborhood $\mathscr{U}_2$ of $\mathscr{N}$, we have $P(\phi_x((x_0, e_0), T)) \leq P(\phi_\xi((\xi_0, \zeta_0), T))$, which was claimed. ∎

Remark 1: In the light of Theorem 3, it appears natural to ask for conditions to let $\mathscr{U}_2$ be arbitrarily large. It turns out that the conditions of Theorem 2 are sufficient to let the statement of Theorem 3 hold true for every $\mathscr{U}_2 = \mathscr{U}_P^{\gamma} \times \mathbb{R}^n$ such that $\mathscr{U}_P^{\gamma} \subset \mathscr{U}_1$, along the lines of the proof of Theorem 2.

Remark 2: To let k_0 suffice (23), or rather to compute an upper bound for k_0, it is possible to overestimate $\frac{1}{2}\phi_e((x_0, e_0), T)^2$ by $V(x_0, e_0)$, as long as we choose $k_1 > \max\{q, \left(\frac{q+1}{2\lambda}\right)^2\}$, since the latter guarantees that V decreases along (ϕ_x, ϕ_e).

VII. Navigation in Unknown Terrain

In navigation problems, particularly in obstacle avoidance problem, it is desirable that the positions ϕ_x follow the solution of the gradient system $\dot{x} = -k\nabla\phi(x)$ (cf. [3]), where the so-called navigation function $P = \varphi$ is designed such that it has its minimum 0 at the target submanifold $\mathscr{M}$ and its maximum 1 at the obstacles $\mathscr{O}$. For our example, we choose the obstacle $\mathscr{O} = \{x \in \mathbb{R}^2 | (x_1 - \pi)^2 + (x_2 + \pi)^2 = 0.25\}$ and the target submanifold $\mathscr{M} = \{x \in \mathbb{R}^2 | x_1 = 0\}$. The resulting navigation function is

$$\varphi(x) = x_1^2 \left(x_1^4 + (x_1 - \pi)^2 + (x_2 + \pi)^2 - 0.25\right)^{-1/2}$$

and we want to let the fully actuated vehicle

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ \frac{1}{m}(F - g\nabla E(x)) \end{bmatrix} \tag{24}$$

with mass m, gravity g, potential energy E, navigate such that its positions approach $\mathscr{M}$ whilst avoiding $\mathscr{O}$. Without losing generality, let $m = 1$, $g = 1$. More, let the potential energy E be determined by the unknown height map

$$E(x) = H(x) = h(1 - \cos(x_2))(1 - \cos(x_1)). \tag{25}$$

Let F be given by the backstepping control law

$$F = -\left(1 + k^2\right)\nabla\varphi(x) - kv - k\nabla^2\varphi(x)v, \tag{26}$$

aiming to let ϕ_x follow solutions of $\dot{x} = -k\nabla\phi(x)$ for $h = 0$ (flat terrain). We however assume that the real value for h is $h = 10$. Consequently, we simulate three scenarios in MATLAB using `ode45` . First, we simulate the system without drift (flat terrain), i.e. $h = 0$ and $k = 1$; second, we simulate the system with drift, i.e. $h = 10$ and $k = 1$; third, we simulate the system with drift and adjusted control gain, i.e. $h = 10$ and $k = 10$. Fig. 1 depicts a plot of $\varphi \circ \phi_x$ versus t for all three scenarios and Fig. 2 depicts a plot of the solution ϕ_x in the (x_1, x_2) plane for all three scenarios. In the first scenario, ϕ_x approaches $\mathscr{M}$ and avoids $\mathscr{O}$, which shows that the controller is designed appropriately for the system without drift. In the second scenario, ϕ_x approaches $\mathscr{O}$ (i.e. collides with the obstacle) due to the drift vector field. In the third scenario, ϕ_x approaches $\mathscr{M}$ and avoids $\mathscr{O}$ despite the drift vector field. Moreover, for all $t \geq 0$, we have $P \circ \phi_x \leq P \circ \phi_\xi$, so that it is possible to assess the navigation function of the system with drift by means of the navigation function of the system without drift.

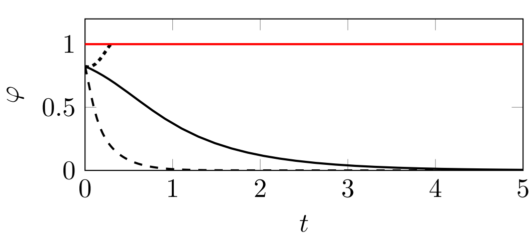

Fig. 1. Plot of the value of $\varphi(\phi_x((x_0, e_0), t))$ for $h = 10$, $k = 10$ (– – –), $h = 10$, $k = 1$ (- - -), $h = 0$, $k = 1$ (——), and $\varphi(x)$ for x being at the boundary of the obstacle (——).

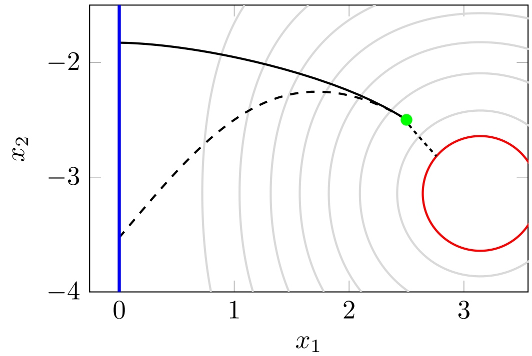

Fig. 2. Plot of the solution $\phi_x((x_0, e_0), t)$ for $h = 10$, $k = 10$ (– – –), $h = 10$, $k = 1$ (- - -), $h = 0$, $k = 1$ (——) together with initial condition x_0 (●), obstacle (○), target submanifold $\mathscr{M}$ (——), and level sets of H (——).

VIII. Conclusion

We considered control problems for mechanical systems whose positions are ought to follow the solution to some gradient system for an appropriately chosen navigation function vanishing on a submanifold. In particular, we were considering the case where the closed loop is designed via backstepping whilst ignoring an unknown vector field governing the velocities, such that the vector field of the closed loop contains a drift vector field. We studied whether or not it is possible to guarantee convergence and stability for the system with drift. We showed that for locally Lipschitz drift vector fields and mild assumptions on the navigation function, the answer is affirmative and merely necessitated gain tuning. Moreover, we showed that it is always possible to overestimate the navigation function of the system with drift by the the navigation function of the system without drift, which has natural application in obstacle avoidance problems. Accordingly, we illustrated our findings with an obstacle avoidance problem in an unknown height map.

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