On the Observability Properties of Systems with Rolling Shutter

Jan Maximilian Montenbruck, Shen Zeng, Frank Allgöwer

***Abstract*— We pose the image reconstruction problem for pictures distorted by rolling shutter as an observability problem. In particular, we study the observability properties of linear systems of whom measurements are taken with a pixel-by-pixel evaluation. We do not concentrate on the technical process behind rolling shutter, but introduce and study it as a systems theoretic property. Assuming recurrences in the time series of pictures which we aim to reconstruct, we derive resonance-like conditions for observability.**

I. Motivation

In digital imaging, active pixel sensors have become a popular alternative to charge coupled devices for their advantages in "bloom" (i.e. having pixels nearby a light source overexposed), power consumption, lag, manufacturing, onboard image processing, scalability, and, most significantly, (monetary) cost. Yet, those active pixel sensors make use of complementary metal-oxide-semiconductors [1], which are a source for "rolling shutter" (as opposed to "global shutter"), a term referring to a pixel-by-pixel, or line-by-line readout of pixel data.

Fig. 1. Distortion effects caused by rolling shutter [2]

The authors are with the Institute for Systems Theory and Automatic Control, University of Stuttgart and thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/2) at the University of Stuttgart. For correspondence, `mailto:montenbruck@ist.uni-stuttgart.de`

If pictures (or time series thereof) are recorded subject to such rolling shutter, distortion effects may occur. The following distortion effects are regularly noticed: "wobble" a.k.a. "jello" (caused by moving the camera), "skew" (caused by an object translating through the picture), "smear" (caused by rotary motion of an object in the picture), and "partial exposure" (caused by varying brightness during recording). Two of these distortions, viz. "smear" and "skew" are depicted in the top and bottom, respectively, of Fig. 1 (taken from [2]).

In the present paper, we ask whether "true" pictures (or time series thereof) without such distortion effects can be reconstructed from the pictures recorded with rolling shutter. We assume that the pictures are generated by a linear dynamical system, thereby emphasizing systems theoretical aspects of that reconstruction problem, i.e.: we pose the image reconstruction problem for pictures recorded subject to rolling shutter in terms of an observability problem.

Our motivation is similar to the one of [3], but we try to reconstruct the information which is lost in measurements by virtue of a systems theoretical approach whereas the authors of [3] interpolate the measurement data and then generate velocity estimates therefrom (using the techniques from [4]).

II. Introduction

We study linear dynamical systems of the form

$$x_{k+1} = Ax_k, \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, of which the measurements

$$\begin{aligned}
y_1 &= \ x_{1,1} && (2)\\
y_2 &= (x_{1,1}, x_{2,2}) && (3)\\
y_3 &= (x_{1,1}, x_{2,2}, x_{3,3}) && (4)\\
&\vdots \\
y_n &= (x_{1,1}, x_{2,2}, x_{3,3}, \dots, x_{n,n}) && (5)\\
y_{n+1} &= \ x_{n+1,1} && (6)\\
y_{n+2} &= (x_{n+1,1}, x_{n+2,2}) && (7)\\
&\vdots
\end{aligned}$$

are taken, wherein $x_{k,j}$ denotes the jth entry of x_k. If x_k is the picture under scrutiny at time k and $x_{k,j}$ is the value of its jth pixel, with n overall pixels in the picture, then this system models the premises of cameras with rolling shutter, where pixel by pixel (or line by line) is evaluated, and thus each pixel is associated with a different time instance of the system the camera shall observe. This premise is depicted in Fig. 2.

Therein, a picture, say an advertisement on an electric display, changes as time proceeds, as illustrated in the left column of Fig. 2. In the right column, the picture is recorded with a rolling shutter camera, i.e. pixel by pixel is evaluated, and hence every pixel belongs to a different time instance of the dynamical system describing the evolution on the left-hand side. Here, A would be a permutation matrix and x_k would be the vector containing integers, each of which would be associated with one of the 9 portions of the picture, "pixels", at time k. In an observability problem, our goal is to reconstruct the (complete) initial condition x_0 of (1) from the measurements (2)-(7). Knowing A, this is equivalent to asking whether we can reconstruct the pictures on the left of Fig. 2 from the measurements on the right.

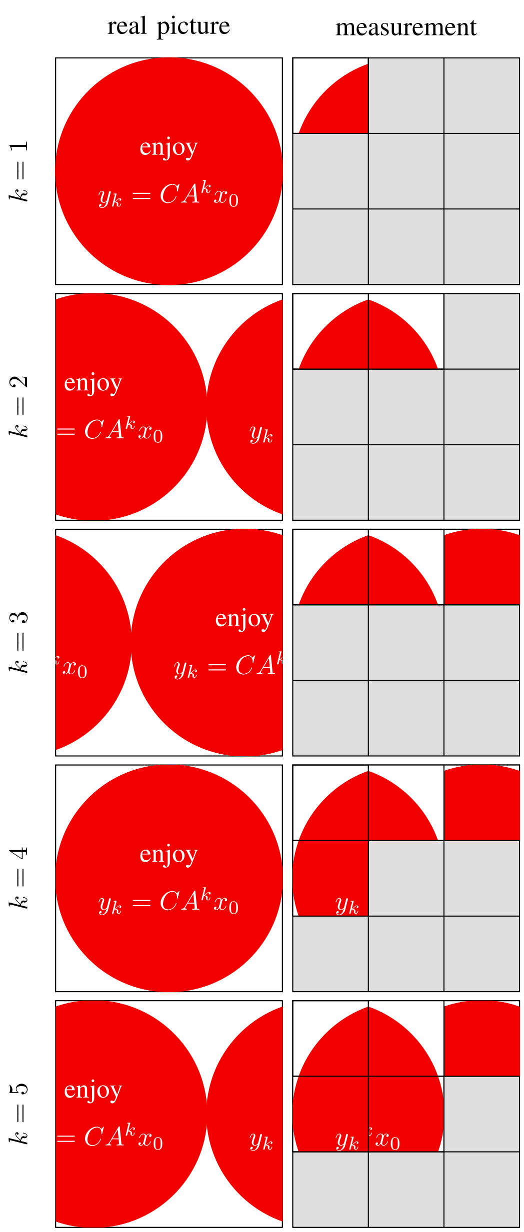

Fig. 2. Rolling shutter effect

III. Problem Statement

We study the observability problem for systems of the form (1) with measurements of the form (2)-(7), i.e. to reconstruct the initial condition x_0 of our dynamical system from the data $(y_k)_{k\in\mathbb{N}}$. Thereby, we do not lose generality if we model our measurement by

$$y_k = e_{1+(k-1) \bmod n} \cdot x_k, \tag{8}$$

wherein $e_i \in \mathbb{R}^n$ denotes the ith vector of the standard basis, as its information content is the same as in (2)-(7).

In principle, as combination of (8) and (1) yields $y_k = e_{1+(k-1) \bmod n} \cdot A^k x_0$, by virtue of a Kalman test, we would have to study the sequence

$$\left(A^{k\top} e_{1+(k-1) \bmod n}\right)_{k\in\mathbb{N}} \tag{9}$$

and ask whether it contains n linearly independent vectors. This condition is necessary and sufficient for observability. However, since $k \mapsto e_{1+(k-1) \bmod n}$ is n-periodic, a rather strong result from [5] states that it is enough to only consider the first n^2 iterations of (9), i.e.

$$A^\top e_1,\ A^{2\top} e_2,\ \ldots,\ A^{n\top} e_n,\ A^{(n+1)\top} e_1,\ \ldots,\ A^{n^2\top} e_n.$$

If those n^2 vectors do not span $\mathbb{R}^n$, then the entire sequence (9) does neither. For the sake of completeness, we briefly repeat this result here: consider a linear system of the form (1) with output

$$y_k = C_k x_k, \tag{10}$$

where $C_{k+s} = C_k$ is periodic for some minimal period s. In other words, the output of the system is produced by different output matrices $C_0, C_1, \ldots, C_s$ that are switched through periodically. If we further assume that this discrete-time system is the result of periodically sampling some continuous-time system, then we can take A to be invertible. Now, group the matrices in the family $(C_k A^k)_{k\in\mathbb{N}}$ with respect to the modes C_i of the output matrix. This results in the grouped block-matrices

$$\begin{pmatrix} C_0 \\ C_0 A^s \\ C_0 A^{2s} \\ \vdots \end{pmatrix}, \begin{pmatrix} C_1 \\ C_1 A^s \\ C_1 A^{2s} \\ \vdots \end{pmatrix} A, \ldots, \begin{pmatrix} C_s \\ C_s A^s \\ C_s A^{2s} \\ \vdots \end{pmatrix} A^s.$$

Since A is assumed to be invertible, the powers of A which are multiplied from the right need not be further considered in the observability analysis. Thus, the span of the sequence $(C_k A^k)_{k\in\mathbb{N}}$ only depends on the cyclic subspaces generated by the pairs $(A^s, C_0), (A^s, C_1), \ldots, (A^s, C_s)$. By Cayley-Hamilton, each cyclic subspace is spanned by the observability matrices of the pairs (A^s, C_i). More explicitly, the only relevant matrices are those in

$$\begin{pmatrix} C_0 \\ C_0 A^s \\ \vdots \\ C_0 A^{(n-1)s} \end{pmatrix}, \begin{pmatrix} C_1 \\ C_1 A^s \\ \vdots \\ C_1 A^{(n-1)s} \end{pmatrix}, \ldots, \begin{pmatrix} C_s \\ C_s A^s \\ \vdots \\ C_s A^{(n-1)s} \end{pmatrix}.$$

The total number of these matrices is precisely $n(s+1)$. A trivial sufficient condition for the switched linear system (10) to be observable is that one of the pairs (A^s, C_i) is observable. Later below, however, we will find that this condition will not be met for models of evolutions of pictures, cf. Fig. 2.

The above considerations for systems with periodic C_i are algebraic, yet generic. In the present paper, we aim to find constructive conditions for observability by exploiting systems theoretic properties of A. We thereby focus on systems that are "natural" models for describing the evolution of pictures, i.e. systems whose solutions neither converge nor diverge but have certain "recurrences", again cf. Fig. 2. Specifically, we derive resonance-like conditions for observability.

It shall be remarked that state-space approaches to image processing have been pursued before [6]. In fact, efforts have been taken to provide general realizations of restoration and recovery filters given as transfer functions (for a broad introduction to these filters, cf. [7, Chapters 6 & 7]) in state-space [8].

However, stochastic (viz. Bayesian) methods for image restoration still remain most popular [9].

IV. Systems Evolving under Permutations

The case where A is a permutation matrix shall receive particular attention. For instance, the dynamical system governing the evolution of the picture in the left column of Fig. 2 evolves under repeated application of some permutation.

In particular, let σ be a permutation, i.e. an injection from $\{1,\ldots,n\}$ onto itself, and let $P_\sigma \in \mathbb{R}^{n\times n}$ be the corresponding permutation matrix, i.e. the matrix whose entry $\sigma(i)$ in row i is 1 but whose other entries are all zero. Reconsider (1) and let $A = P_\sigma$. Then our measurements (8) amount to

$$\begin{aligned} y_1 &= e_1 \cdot P_\sigma x_0 \\ &= x_{0,\sigma(1)} && (11)\\ y_2 &= e_2 \cdot P_\sigma^2 x_0 = e_2 \cdot P_{\sigma^2} x_0 \\ &= x_{0,\sigma^2(2)} && (12)\\ y_3 &= e_3 \cdot P_\sigma^3 x_0 = e_2 \cdot P_{\sigma^3} x_0 \\ &= x_{0,\sigma^3(3)} && (13)\\ &\vdots \\ y_n &= e_n \cdot P_\sigma^n x_0 = e_n \cdot P_{\sigma^n} x_0 \\ &= x_{0,\sigma^n(n)} && (14)\\ y_{n+1} &= e_1 \cdot P_\sigma^{n+1} x_0 = e_1 \cdot P_{\sigma^{n+1}} x_0 \\ &= x_{0,\sigma^{n+1}(1)} && (15)\\ y_{n+2} &= e_2 \cdot P_\sigma^{n+2} x_0 = e_2 \cdot P_{\sigma^{n+2}} x_0 \\ &= x_{0,\sigma^{n+2}(2)} && (16)\\ &\vdots \end{aligned}$$

whence we can (only) reconstruct the entries $\sigma^{in+k}(k)$, $i \in \mathbb{N}$, of x_0. This leads us to the following proposition, asking for $\sigma^{in+k}(k)$ to cover all indices of x_0.

Theorem 1: Let σ be a permutation of $\{1,\ldots,n\}$ and in (1), let $A = P_\sigma \in \mathbb{R}^{n\times n}$ be the corresponding permutation matrix. Then x_0 can be reconstructed from $(y_k)_{k\in\mathbb{N}}$ if and only if

$$\mathbb{N}\times\{1,\ldots,n\} \to \{1,\ldots,n\}, \quad (i,k) \mapsto \sigma^{in+k}(k) \tag{17}$$

is onto.

Proof: Reconsider the iteration (11)-(16) and find that every entry of x_0 is attained by $k \mapsto y_k$ if and only if $(i,k) \mapsto \sigma^{in+k}(k)$ is surjective. ■

Example 1: Let $n = 5$. In cycle notation, consider the permutation

$$(4\ 2\ 3\ 1)\ (5) \tag{18}$$

for which we have

$$\begin{aligned} 1 &= \sigma^2(2), & x_{0,1} &= y_2 && (19)\\ 2 &= \sigma^3(3), & x_{0,2} &= y_3 && (20)\\ 3 &= \sigma^8(3), & x_{0,3} &= y_8 && (21)\\ 4 &= \sigma^1(1), & x_{0,4} &= y_1 && (22)\\ 5 &= \sigma^5(5), & x_{0,5} &= y_5 && (23) \end{aligned}$$

whence $(i,k) \mapsto \sigma^{in+k}(k)$ is onto $\{1,\ldots,n\}$ and the system (1) with measurements (2)-(7) is observable for $A = P_\sigma$. In particular, as argued above, x_0 can be reconstructed from y_1, y_2, y_3, y_5, y_8. If we had $n = 4$ and deleted the cycle $\sigma(5) = 5$, then the system would not be observable, for $\sigma^{in+k}(k) = \sigma^k(k)$ whence $\sigma^{in+k}(k)$ could not attain the value 3. The reason for this is that the cycle length is n.

Our previous condition has a particularly insightful interpretation in terms of cycles of σ. Specifically, the last example illustrates that permutations in which n is a multiple of a cycle length deserve particular attention.

Proposition 1: Let σ be a permutation of $\{1,\ldots,n\}$ and in (1), let $A = P_\sigma \in \mathbb{R}^{n\times n}$ be the corresponding permutation matrix. Let all cycles of σ have lengths such that n is a multiple of these lengths. Then x_0 can be reconstructed from $(y_k)_{k\in\mathbb{N}}$ if and only if $k \mapsto \sigma^k(k)$ is injective.

Proof: Reconsider Theorem 1. Since distinct cycles are disjoint, it is sufficient to restrict our attention to one particular cycle, say of length ℓ, with n being a multiple of ℓ. Thus, for all $i \in \mathbb{N}$, in is a multiple of ℓ, as well, and hence $(i,k) \mapsto \sigma^{in+k}(k)$ is surjective if and only if $k \mapsto \sigma^k(k)$ is surjective. As surjectivity is equivalent to injectivity on finite sets, this completes the proof. ■

Example 2: The previous proposition can be applied to our example from Fig. 2. In particular, numbering the $n = 9$ portions of the picture, "pixels", row-wise, the permutation σ governing the evolution of the picture in the left column of the figure decomposes into the cycles

$$(1\ 2\ 3)\ (4\ 5\ 6)\ (7\ 8\ 9) \tag{24}$$

which all have length 3, of which $n = 9$ is a multiple. Now consider the cycle $(1\ 2\ 3)$. We have that $\sigma(1) = 2$, $\sigma^2(2) = 1$, and $\sigma^3(3) = 3$, whence $k \mapsto \sigma^k(k)$ is (both injective and) surjective and thus the pictures on the left can be reconstructed from those on the right.

Specifically, x_0 can be inferred from $y_1, \dots, y_9$. In other words, system (1) with measurements (2)-(7) is observable with $A = P_\sigma$. On the contrary, should each row have four pixels, amounting to a total of $n = 12$ pixels in the picture, then the permutation governing the evolution of the picture would decompose into the cycles

$$(1\ 2\ 3\ 4)\ (5\ 6\ 7\ 8)\ (9\ 10\ 11\ 12)\,. \tag{25}$$

These cycles all have length 4, of which $n = 12$ is, again, a multiple. As before, we may restrict our attention to the first cycle $(1\ 2\ 3\ 4)$. Here, we find that $\sigma^2(2) = 4$, but $\sigma^4(4)$ is 4, as well, whence our dynamical system cannot be observable by virtue of our foregoing proposition. In general, for an even number of pixels per row ℓ, we will always have that

$$\sigma^{\ell/2}(\ell/2) = \sigma^{\ell}(\ell) = \ell \tag{26}$$

and hence arrive at an unobservable system. Similarly, for an odd number of pixels per row, the system will always be observable. We also find that, in order to have $k \mapsto y_k$ attain all values of x_0 at least once (and hence bringing us into the position to reconstruct x_0), we would have have to wait for $k = n$ time instances, which is just the number of pixels in the picture.

V. Systems Evolving under Rotations

In this section, the case where A is a rotation matrix shall receive particular attention. As for the permutation matrices, a reason to consider rotations is that iterated applications thereof carry no nonzero vector to zero, i.e. these iterations exhibit nontrivial behavior. In particular, let $n = 2$ and

$$A = R_\alpha = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{27}$$

for which we receive the measurements

$$\begin{aligned} y_1 &= e_1 \cdot R_\alpha x_0 \\ &= x_{0,1}\cos(\alpha) - x_{0,2}\sin(\alpha) \qquad (28)\\ y_2 &= e_2 \cdot R_\alpha^2 x_0 = e_2 \cdot R_{2\alpha} x_0 \\ &= x_{0,1}\sin(2\alpha) + x_{0,2}\cos(2\alpha) \qquad (29)\\ y_3 &= e_1 \cdot R_\alpha^3 x_0 = e_1 \cdot R_{3\alpha} x_0 \\ &= x_{0,1}\cos(3\alpha) - x_{0,2}\sin(3\alpha) \qquad (30)\\ y_4 &= e_2 \cdot R_\alpha^4 x_0 = e_2 \cdot R_{4\alpha} x_0 \\ &= x_{0,1}\sin(4\alpha) + x_{0,2}\cos(4\alpha) \qquad (31)\\ &\vdots \end{aligned}$$

and thus find that x_0 is related to these measurements through

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} \cos(1\alpha) & -\sin(1\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \\ \cos(3\alpha) & -\sin(3\alpha) \\ \sin(4\alpha) & \cos(4\alpha) \\ \vdots & \vdots \end{pmatrix} x_0. \tag{32}$$

This leads us to the following proposition.

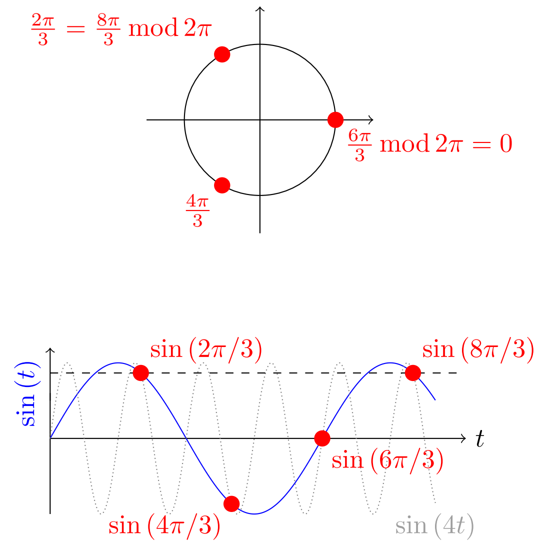

Fig. 3. Resonance $4\frac{2\pi}{3} \bmod 2\pi = \frac{2\pi}{3}$

Proposition 2: Let $\alpha \in (0, 2\pi)$ and in (1), let $A = R_\alpha$ be the corresponding rotation matrix. Then x_0 can be reconstructed from $(y_k)_{k\in\mathbb{N}}$ if

$$\exists p, q \in \mathbb{N}: \quad ((2p+1)\alpha) \bmod 2\pi = (2q\alpha) \bmod 2\pi. \tag{33}$$

Proof: Reconsider the relation (32) and recall that rotation matrices are invertible. If an even multiple $2q$ and an odd multiple $2p+1$ of α are the same, modulo 2π, then the rows $2p+1$ and $2q$ of the matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$ form a rotation matrix, whence (32) admits a unique solution in x_0. ■

Example 3: Let ($n = 2$ and) α be $2\pi/3$. Then

$$4\alpha \bmod 2\pi = \alpha$$

and hence the rows 4 and 1 of the matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$ in (32) are themselves a rotation matrix, i.e. x_0 can be reconstructed from y_1 and y_4, from which we infer that the system (1) with measurements (2)-(7) is observable with $A = R_\alpha$. Two different interpretations of this resonance $4\alpha \bmod 2\pi = \alpha$ are depicted in Fig. 3. On the other hand, let $\alpha = \pi$. Although the condition (33) cannot be satisfied, for any even multiple of π is zero modulo 2π and every odd multiple of π is π modulo 2π, the rows 1 and 2 of the matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$ in (32) are $(-1, 0)$ and $(0, 1)$, respectively, such that x_0 may still be reconstructed (viz. from y_1 and y_2), and illustrating that the condition from Proposition 2 is not necessary.

Though stemming from the particular case that $n = 2$, the observation from Proposition 2 offer to generalize to the case where A is from some arbitrary matrix (Lie) group G, say the orthogonal group $\mathrm{O}(n)$, or the special linear group $\mathrm{SL}(n)$, as the following proposition formalizes.

Theorem 2: In (1), let $A = G \in \mathrm{G}$ for some matrix group G. Then x_0 can be reconstructed from $(y_k)_{k\in\mathbb{N}}$ if

$$\exists G' \in \mathrm{G}: \quad \forall k = 1, \dots, n \quad \exists i \in \mathbb{N}: \quad G^{in+k} = G'. \tag{34}$$

Proof: The initial condition x_0 is related to $(y_k)_{k\in\mathbb{N}}$ through

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \end{pmatrix} = \begin{pmatrix} e_1^\top G \\ e_2^\top G^2 \\ \vdots \\ e_n^\top G^n \\ e_1^\top G^{n+1} \\ e_2^\top G^{n+2} \\ \vdots \end{pmatrix} x_0. \tag{35}$$

If the condition (34) is met, then some row $in + k$ of the matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$, for which $G^{in+k} = G'$, is precisely the kth row of G'. As we insisted that this condition shall be satisfied for all $k = 1, \dots, n$, we may find all rows of G' within our matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$. Since members of G must be invertible, this also reveals that (35) admits a unique solution in x_0. ■

Example 4: Let $n = 3$ and consider the special linear group $\mathrm{G} = \mathrm{SL}(3)$, of whom

$$G = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ -3/2 & 3/2 & 1 \end{pmatrix} \tag{36}$$

is a member. Consider another member,

$$G' = \begin{pmatrix} 5/2 & 1/2 & -2 \\ 1/2 & 1/2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \tag{37}$$

of $\mathrm{SL}(3)$. We have that

$$G^{2n+1} = G', \quad y_{2n+1} = e_1^\top G' x_0 \tag{38}$$

$$G^{3n+2} = G', \quad y_{3n+2} = e_2^\top G' x_0 \tag{39}$$

$$G^{4n+3} = G', \quad y_{4n+3} = e_3^\top G' x_0 \tag{40}$$

letting us conclude that the condition (34) is satisfied and that the system (1) with measurements (2)-(7) is thus observable for $A = G$. In particular, x_0 can be reconstructed from

$$\begin{pmatrix} y_7 \\ y_{11} \\ y_{15} \end{pmatrix} = G' x_0, \qquad x_0 = G \begin{pmatrix} y_7 \\ y_{11} \\ y_{15} \end{pmatrix} \tag{41}$$

via inversion of G', which was just G^{-1}.

VI. Systems with Periodic Orbits

The last result of the previous section, i.e. Theorem 2, offers to extend to rather general systems which admit periodic orbits. In particular, let A be such that

$$\exists p \in \mathbb{N}: \quad \forall x_0, \quad A^p x_0 = x_0, \tag{42}$$

i.e. let (1) posses a periodic orbit with period p for any initial condition. This amounts to requiring that A is of finite order p.

Theorem 3: In (1), let $A^p = I_n$, the $n \times n$ identity matrix, for some p. Then x_0 can be reconstructed from $(y_k)_{k\in\mathbb{N}}$ if

$$\mathbb{N} \times \mathbb{N} \to \mathbb{Z}, \quad (i, j) \mapsto ip - jn \tag{43}$$

is onto $\{1, \dots, n\}$, i.e., in particular, if p and n are coprime.

Proof: The initial condition x_0 is related to $(y_k)_{k\in\mathbb{N}}$ through

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ y_{n+2} \\ \vdots \end{pmatrix} = \begin{pmatrix} e_1^\top A \\ e_2^\top A^2 \\ \vdots \\ e_n^\top A^n \\ e_1^\top A^{n+1} \\ e_2^\top A^{n+2} \\ \vdots \end{pmatrix} x_0 \tag{44}$$

wherein $A^{ip} = I_n$ for all $i \in \mathbb{N}$. Now if (43) is onto $\{1, \dots, n\}$, then

$$\forall k = 1, \dots, n \quad \exists i, j \in \mathbb{N}: \quad k + jn = ip \tag{45}$$

and hence the matrix relating x_0 to $(y_k)_{k\in\mathbb{N}}$ in (44) contains the rows $e_1^\top, \dots, e_n^\top$. Having found n linearly independent rows, we infer that (44) admits a unique solution in x_0. ■

Example 5: Let $n = 4$. Following the construction in [10], the matrix

$$A = \begin{pmatrix} 1 & 3/2 & 3/2 & -\sin(2\pi/3) \\ 0 & 1 & 0 & 0 \\ 0 & -3/2 & -1/2 & \sin(2\pi/3) \\ 0 & -\sin(2\pi/3) & -\sin(2\pi/3) & -1/2 \end{pmatrix} \tag{46}$$

is finite of order $p = 3$, i.e. $A^3 = I_4$. We have that

$$1 = 3p - 2n, \quad x_{0,1} = y_{2n+1} \tag{47}$$

$$2 = 2p - 1n, \quad x_{0,2} = y_{1n+2} \tag{48}$$

$$3 = 5p - 3n, \quad x_{0,3} = y_{3n+3} \tag{49}$$

$$4 = 4p - 2n, \quad x_{0,4} = y_{2n+4} \tag{50}$$

and hence infer that (43) is onto $\{1, \dots, n\}$, letting our system (1) with measurements (2)-(7) remain observable. In particular, as argued above, x_0 can be reconstructed from y_6, y_9, y_{12}, y_{15}.

VII. Application: Removing Smear

In the present section, we now apply the understanding gained from our results to the distortion effect from the top of Fig. 1, i.e. the "smear". In Fig. 4, we depict a simple sequence of pictures on the left in which 5 black pixels among $n = 25$ overall pixels rotate and thus cause a smear in the pixel-by-pixel measurements depicted on the right. The behavior on the left is modeled by a system evolving under the permutation

$$(11\ 21\ 23\ 25\ 15\ 5\ 3\ 1)\ (12\ 17\ 18\ 19\ 14\ 9\ 8\ 7),$$

in cycle notation, with all indices not explicitly mentioned being 1-cycles. The resulting permutation matrix has finite order $p = 8$. As the permutation matrices are also a (finite) matrix group, we can here make use of either Theorem 1, 2, or 3. One is tempted to think that Theorem 1 is preferable (for being both sufficient and necessary) but we here opt to apply Theorem 3 for the following reason: in order to verify that (43) is onto $\{1, \dots, n\}$, it is enough to only know p and n. In particular, it is not required to know A explicitly.

The same statement remains true if we ask to reconstruct x_0, i.e. one does not have to know A explicitly to do so, but it is sufficient to know p. More particular, we have

$$1 = 176 - 175 = 22p - 7n \tag{51}$$
$$2 = 152 - 150 = 19p - 6n \tag{52}$$
$$3 = 128 - 125 = 16p - 5n \tag{53}$$
$$4 = 104 - 100 = 13p - 4n \tag{54}$$
$$\vdots$$
$$25 = 200 - 175 = 25p - 7n \tag{55}$$

and, in general, $k = ip - jn$ with

$$j = p - 1 - (k-1) \bmod p \tag{56}$$

hence knowing that (43) is onto $\{1, \ldots, n\}$, and, further, letting us conclude that the kth pixel of the picture x_0 can be reconstructed via

$$x_{0,k} = y_{n(p-1-(k-1) \bmod p)+k}. \tag{57}$$

For instance, the $k = 8$th pixel, i.e. the third pixel of the second row, is correctly identified as being white in x_0.

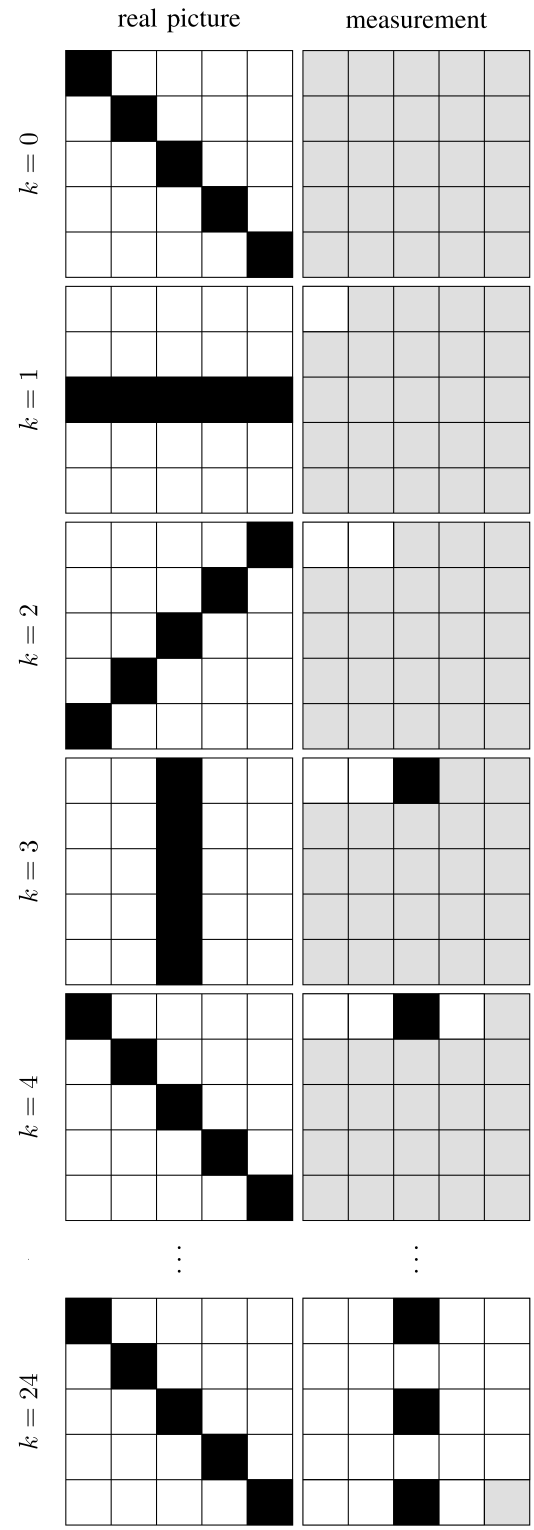

Fig. 4. Rotating object and "smear"

VIII. Conclusion and Outlook

We posed the problem of reconstructing a time series of pictures from measurements with rolling shutter, i.e. with a pixel-by-pixel (or line-by-line) evaluation, as an observability problem. Assuming certain recurrences of the dynamical system governing the evolution of the pictures, we derived resonance-like conditions for observability.

Among other examples, we illustrated our findings on the removal of a distortion effect called "smear" which is known to arise in cameras with rolling shutter.

In the future, it would be interesting to ask for unobservable subspaces occurring in this setting, i.e. circumstances under which a picture can only be partially reconstructed. Further, since the reconstruction methods we presented all relied on knowing the recurrences of the dynamical system governing the evolution of pictures, it should be of interest how precisely a picture can be reconstructed if we are only provided estimates of those recurrences.

References

[1] E. R. Fossum, "CMOS image sensors: Electronic camera-on-a-chip," *IEEE Transactions on Electron Devices*, vol. 44, no. 10, pp. 1689–1698, 1997.

[2] `commons.wikimedia.org`, licensed by *CC BY-SA 3.0*.

[3] C.-K. Liang, L.-W. Chang, and H. H. Chen, "Analysis and compensation of rolling shutter effect," *IEEE Transactions on Image Processing*, vol. 17, no. 8, pp. 1323–1330, 2008.

[4] J. Y. A. Wang and E. H. Adelson, "Representing moving images with layers," *IEEE Transactions on Image Processing*, vol. 3, no. 5, pp. 625–638, 1994.

[5] S. Bittanti, P. Colaneri, and G. De Nicolao, "Discrete-time linear periodic systems: A note on the reachability and controllability interval length," *Systems & Control Letters*, vol. 8, pp. 75–78, 1986.

[6] R. Roesser, "A discrete state-space model for linear image processing," *IEEE Transactions on Automatic Control*, vol. 20, no. 1, pp. 1–10, 1975.

[7] A. Rosenfeld and A. C. Kak, *Digital Picture Processing* (Second Edition, Volume 1). Academic Press, 1982.

[8] E. Fornasini and G. Marchesini, "State-space realization theory of two-dimensional filters," *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 484–492, 1976.

[9] S. Geman and D. Geman, "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 6, no. 6, pp. 721–741, 1984.

[10] R. Koo, "A classification of matrices of finite order over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$," *Mathematics Magazine*, vol. 76, no. 2, pp. 143–148, 2003.