

# Pinning Capital Stock and Gross Investment Rate in Competing Rationally Managed Firms <sup>★</sup>

Jan Maximilian Montenbruck and Frank Allgöwer

*Institute for Systems Theory and Automatic Control, University of Stuttgart,  
Pfaffenwaldring 9, 70550 Stuttgart, Germany*

---

**Abstract:** We consider two competing firms in a competitive industry, modeled as proposed by J. P. Gould. Assuming that one company is a start-up and the other company is a well-established one with a dynamic behavior determining the optimum, we take the point of view of the start-up. We propose a pinning control scheme that makes the solutions of the start-up in capital stock and gross investment rate approach the solutions of the company determining the desired behavior. The approach is discussed based on numerical examples. An extension of the proposed control law equips it with the capability to cope with uncertainties. We interpret our findings from the perspectives of control theory and economics.

---

## 1. INTRODUCTION

### 1.1 Motivation

If one wants to apply control theory to improve the performance of a business in a competitive industry, one usually uses techniques from optimal control to maximize the value of all future cash flows. In doing so, the optimum is determined by the cost functional, which is chosen subjectively. If, in contrast, one lets the optimum be determined by the performance of another company, different control theoretical techniques are necessary.

For doing so, we propose a pinning controller that forces the gross investment rate and the capital stock of a company to converge to the gross investment rate and capital stock of another company exponentially.

### 1.2 Previous Work

To derive our control laws, we utilize the model of a rationally managed firm in a competitive industry proposed by Gould [1968], who applied optimal control theory to maximize the value of all future cash flows. Moreover, for our controller design procedure, we exploit the QUAD property of the underlying vector field, which has in the past been proposed by DeLellis et al. [2011, 2009, 2008] for similar problems. Vector fields with this property are particularly suited to derive pinning controllers, such as by Grigoriev et al. [1997], Li et al. [2004], Chen et al. [2007], Porfiri and di Bernardo [2008].

Classical control laws for microeconomical systems are derived with methods from optimal control; a finite-time profit-maximizing controller was designed by Simaan and Takayama [1976] considering monopolists. A review on feedback controllers assisting business decisions was written by Morecroft [1985] and a broad insight on such techniques is given in Morecroft [2007]. A more dynamics- and less feedback-related point of view on businesses is elaborated in Serman [2000].

---

<sup>★</sup> Corresp. author: J.M. Montenbruck (jan-maximilian.montenbruck@ist.uni-stuttgart.de). The authors would like to thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart.

Chen and Chen [2007] have shown how to control the chaotic behavior of the Cournot-Puu duopoly model. Equilibria and (optimal) control laws in oligopolies were investigated by Karp and Perloff [1993].

Recently, analyses of micro- and macroeconomical systems has been performed through the point of view of complex networks; cluster synchronization phenomena have been observed in stock markets by Basalto et al. [2005]. Similarly, Strogatz [2001] observed clustering among the largest companies in the US. Hakansson and Ford [2002] discuss how to cope with the phenomena and paradoxa arising when coupling companies in a network. The clustering coefficients of business networks on bipartite graphs were analyzed by Souma et al. [2003].

### 1.3 Contribution and Structure of the Document

Previous work has focused on designing controllers by means of optimal control and on analyzing dynamical phenomena in coupled businesses. In contrast, we want to couple businesses in order to design controllers, i.e. we introduce a coupling between two companies to achieve certain convergence properties for one of the companies. In doing so, we will achieve strong convergence properties.

The remainder of the document is structured as follows; in section 2, we formalize the control problem that we want to solve. Therein, we introduce the model that we presume for our controller design procedure. We assume this model to be valid for both companies. In particular, we will assume one company to be a well-established, well-running company and the other company to be a start-up. The desired behavior for the start-up will thus be given by the behavior of the well-established, well-running company. We will show that the vector fields of the models satisfy the QUAD property, which is often assumed in literature for synchronization and pinning control. In section 3, we present our main result. Subsection 3.1 contains the controller design procedure using a pinning scheme and the proof for the convergence properties of the resulting closed-loop, viz. gross investment rate and capital stock of the start-up converge exponentially towards gross investment rate and capital stock of the well-established, well-running company. In subsection 3.2, we present a method for choosing production

quantity and wage such that the proposed control law can be realized. We discuss the result in section 4 and illustrate it on a numerical example in section 5. In section 6, we propose an extension of our main result for coping with uncertainties. Section 7 concludes the paper.

## 2. PROBLEM STATEMENT

Gould [1968] has proposed the two-dimensional dynamical system

$$\begin{bmatrix} \dot{K}^*(t) \\ \dot{I}^*(t) \end{bmatrix} = \begin{bmatrix} I^*(t) - \delta^*(t) K^*(t) \\ (I^*(t) + q^*)(r(t) + \delta^*(t)) - P^*(t) G^*(t) \end{bmatrix} \quad (1)$$

to model the dynamics of a rationally managed firm in a competitive industry, where  $K^* : \mathbb{R}^+ \rightarrow \mathbb{R}$  is the capital stock,  $I^* : \mathbb{R}^+ \rightarrow \mathbb{R}$  the gross investment rate,  $\delta^* : \mathbb{R}^+ \rightarrow \mathbb{R}$  the percentage of capital stock taken as replacement investment,  $q^* \in \mathbb{R}$  the ratio between the costs associated with investing in capital stock and the gross investment rate,  $r : \mathbb{R}^+ \rightarrow \mathbb{R}$  the instantaneous interest rate,  $P^* : \mathbb{R}^+ \rightarrow \mathbb{R}$  denotes the product price, and  $G^* : \mathbb{R}^+ \rightarrow \mathbb{R}$  is determined by

$$G^* = \frac{\partial}{\partial K^*} F^* \left( \frac{\partial}{\partial L^*} F^* \right)^{-1}, \quad (2)$$

where  $F^* : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is the production quantity and  $L^*$  the labor input.  $G$  and  $\delta$  are decision variables. The initial values of (1) are denoted by  $K^*(0) = K_0^*$  and  $I^*(0) = I_0^*$ .

Traditionally, (1) is used to derive inputs that maximize certain cost functionals quantifying the value of the firm by means of optimal control theory.

We assume that (1) is a well-established, well-running company and that

$$\begin{bmatrix} \dot{K}(t) \\ \dot{I}(t) \end{bmatrix} = \begin{bmatrix} I(t) - \delta(t) K(t) \\ (I(t) + q)(r(t) + \delta(t)) - P(t) G(t) \end{bmatrix} \quad (3)$$

is a start-up (i.e. we have initial conditions  $G(t_0) = G_0 \ll G_0^*$  and  $I(t_0) = I_0 \ll I_0^*$  for (3)), whose point of view we take. Our goal is to design inputs  $\delta$  and  $G$  such that the solutions of (3) converge to the solutions of (1). As  $G$  is only a virtual quantity representing the influence of our production, we will first design  $G$  and later choose  $F$  such that

$$G = \frac{\partial}{\partial K} F \left( \frac{\partial}{\partial L} F \right)^{-1} \quad (4)$$

holds true. In doing so, we assume that the quantities  $K^*$ ,  $I^*$ ,  $P^*$ , and  $\delta^*$  are known to the start-up. We are hence interested in feedbacks  $C$  of the form

$$\begin{bmatrix} \delta(t) \\ G(t) \end{bmatrix} = C(K^*(t), I^*(t), P^*(t), \delta^*(t)) \quad (5)$$

and we will consequently design such herein. Note that we do not want to derive inputs that are to be applied in practice precisely subject to equations, but rather to derive equations that provide support in the actual choice of decision variables.

For doing so, let us define the error

$$E(t) = \begin{bmatrix} K^*(t) - K(t) \\ I^*(t) - I(t) \end{bmatrix} \quad (6)$$

and the error dynamics

$$\dot{E}(t) = \begin{bmatrix} \dot{K}^*(t) - \dot{K}(t) \\ \dot{I}^*(t) - \dot{I}(t) \end{bmatrix} = X(E) \quad (7)$$

and reformulate our design goal to making  $E$  go to zero as time approaches infinity.

For such tasks, i.e. synchronization of trajectories of systems, certain assumptions are frequently imposed in literature. Among them is the QUAD condition studied by DeLellis et al. [2011, 2009, 2008]. Namely, a system  $\dot{x} = f(x)$  is said to be QUAD( $\Delta, \omega$ ), if  $(a-b)^\top (f(a) - f(b)) - (a-b)^\top \Delta (a-b) \leq -\omega (a-b)^\top (a-b)$  for all  $a, b$ , with  $\Delta$  some diagonal matrix and  $\omega$  some finite scalar. If a system is QUAD, it is particularly appealing for application of certain control schemes; e.g., the Lie derivatives of certain Lyapunov functions are easily proven to be negative definite for QUAD systems.

*Claim 1.* The system (3) is QUAD( $\Delta, \omega$ ) for all  $\omega$  and  $\Delta = \Delta_1 \oplus \Delta_2$  satisfying  $4(\Delta_2 - \omega - r(t) - \delta(t))(\Delta_1 - \omega + \delta(t)) = 1$ , where  $\oplus$  is the direct sum of matrices.

**Proof.** First, let us rewrite the QUAD condition for system (3), i.e.

$$\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix}^\top \begin{bmatrix} a_2 - \delta(t)a_1 - b_2 + \delta(t)b_1 \\ (a_2 + q)(r(t) + \delta(t)) - (b_2 + q)(r(t) + \delta(t)) \end{bmatrix} \leq \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix}^\top (\Delta - \omega I_2) \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix}, \quad (8)$$

when using the notation  $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ . The latter is just

$$\begin{aligned} \delta(t)(a_1 - b_1)^2 - (a_1 - b_1)(a_2 - b_2) - (r(t) + \delta(t))(a_2 - b_2)^2 \\ \geq -(\Delta_1 - \omega)(a_1 - b_1)^2 - (\Delta_2 - \omega)(a_2 - b_2)^2 \end{aligned} \quad (9)$$

when factoring out. We consequently have

$$\begin{aligned} 0 \leq (a_1 - b_1)^2 (\Delta_1 - \omega + \delta(t)) + \\ (a_2 - b_2)^2 (\Delta_2 - \omega - r(t) - \delta(t)) - (a_1 - b_1)(a_2 - b_2), \end{aligned} \quad (10)$$

which is satisfied with  $\varepsilon = 2(\Delta_2 - \omega - r(t) - \delta(t))$  and  $1 = 2\varepsilon(\Delta_1 - \omega + \delta(t))$  using Young's inequality. Equating for  $\varepsilon$ , we arrive at  $4(\Delta_2 - \omega - r(t) - \delta(t))(\Delta_1 - \omega + \delta(t)) = 1$ . ■

## 3. MAIN RESULT

### 3.1 Pinning Scheme

As we assume that company (1) does not know what the start-up is currently working on, and as the start-up cannot influence the dynamics of company (1), we have to leave  $\delta^*$  and  $G^*$  untouched. Hence, undirected (i.e. bidirectional) diffusive couplings are no means to solve the posed problem. Instead, we will have to use pinning, i.e. applying a certain input to one node of a network to achieve certain properties for the entire network. Such techniques have thoroughly been studied for QUAD Systems, e.g. by Grigoriev et al. [1997], Li et al. [2004], Chen et al. [2007], Porfiri and di Bernardo [2008]. As such, we propose the formulæ

$$\begin{aligned} C_1(K^*(t), I^*(t), P^*(t), \delta^*(t)) = \\ 1 + \frac{1}{K(t)} (-K^*(t) - I^*(t) + I(t) + \delta^*(t) K^*(t)) \end{aligned} \quad (11)$$

$$\begin{aligned} C_2(K^*(t), I^*(t), P^*(t), \delta^*(t)) = \\ \frac{1}{P(t)} (P^*(t) G^*(t) + (I(t) + q)(r(t) + \delta(t) - 1) - \\ (I^*(t) + q^*)(r(t) + \delta^*(t) - 1) + q - q^*) \end{aligned} \quad (12)$$

and compose the control law (5) according to

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \quad (13)$$

We consequently have the following proposition.

*Claim 2.* Consider the systems (1) and (3) and the feedback (5) under pinning scheme (11), (12), (13). Then the error dynamics (7) are uniformly exponentially stable at the origin.

**Proof.** Consider the Lyapunov function candidate

$$V(E) = \frac{1}{2} E^\top E. \quad (14)$$

Then we clearly have  $\gamma_1 \|E\|^\sigma \leq V(E) \leq \gamma_2 \|E\|^\sigma$  with  $\sigma = 2$  and  $\gamma_1 = \gamma_2 = \frac{1}{2}$  (cf. Khalil [1996]). Letting  $L_X V(E)$  denote the Lie derivative of  $V$  along the vector field  $X$ , where  $X$  is given by (7), we have

$$L_X V(E) = (K^*(t) - K(t)) (\dot{K}^*(t) - \dot{K}(t)) + (I^*(t) - I(t)) (\dot{I}^*(t) - \dot{I}(t)) \quad (15)$$

and we substitute (1) and (3) to arrive at

$$\begin{aligned} L_X V(E) &= (K^*(t) - K(t)) \\ & (I^*(t) - \delta^*(t) K^*(t) - I(t) + \delta(t) K(t)) + \\ & (I^*(t) - I(t)) ((I^*(t) + q^*) (r(t) + \delta^*(t)) - \\ & P^*(t) G^*(t) - (I(t) + q) (r(t) + \delta(t)) + P(t) G(t)). \end{aligned} \quad (16)$$

Substituting the feedback (5) under the pinning scheme (11), (12), (13) yields

$$\begin{aligned} L_X V(E) &= (K^*(t) - K(t)) \\ & (I^*(t) - \delta^*(t) K^*(t) - I(t) + \\ & K(t) - K^*(t) - I^*(t) + I(t) + \delta^*(t) K^*(t)) + \\ & (I^*(t) - I(t)) ((I^*(t) + q^*) (r(t) + \delta^*(t)) - \\ & P^*(t) G^*(t) - (I(t) + q) (r(t) + \delta(t)) + \\ & P^*(t) G^*(t) + (I(t) + q) (r(t) + \delta(t) - 1) - \\ & (I^*(t) + q^*) (r(t) + \delta^*(t) - 1) + q - q^*). \end{aligned} \quad (17)$$

Simplifying, the latter is just

$$L_X V(E) = - (K^*(t) - K(t))^2 - (I^*(t) - I(t))^2, \quad (18)$$

which satisfies  $L_X V(E) \leq -\gamma_3 \|E\|^\sigma$  with  $\gamma_3 = 1$  (cf. Khalil [1996]). According to Lyapunov's direct method, hence, the origin of (7) is uniformly exponentially stable. ■

### 3.2 Choosing Production Quantity and Wage

For the proposed pinning scheme,  $\delta$  and  $G$  have to be chosen subject to formulæ (11) and (12), respectively. While  $\delta$  is a decision variable,  $G$  results indirectly from the production quantity  $F$  according to

$$G = \frac{\partial}{\partial K} F \left( \frac{\partial}{\partial L} F \right)^{-1}, \quad (19)$$

where

$$\frac{K(t)}{L(t)} = \left( \frac{\partial}{\partial L} F \right)^{-1} \left( \frac{W(t)}{P(t)} \right), \quad (20)$$

and  $W$  is the wage. Therein,  $F$  is often assumed to be homogeneous of degree one, i.e. of form

$$F(t, K, L) = \alpha(t) K + \beta(t) L, \quad (21)$$

where  $\beta$  is determined by the labor efficiency (which we cannot influence) and  $\alpha$  is typically determined by production planning (which we can influence). Also, the wage  $W$  can be chosen up to a certain degree. We would thus want to design  $\alpha$  and  $W$ ,

such that we arrive at a function  $F$  that lets  $G$  satisfy (12). For doing so, we take the production planning formula

$$\alpha(t) = P^*(t) G^*(t) + (I(t) + q) (r(t) + \delta(t) - 1) - (I^*(t) + q^*) (r(t) + \delta^*(t) - 1) + q - q^* \quad (22)$$

and the formula

$$W(t) = \frac{K(t) P^2(t)}{L(t)} \quad (23)$$

to determine the wage.

*Claim 3.* If  $\alpha$  and  $W$  are chosen subject to formulæ (22) and (23), then  $G$  satisfies (12).

**Proof.** Taking (21), then  $\left( \frac{\partial}{\partial L} F \right)^{-1} = \frac{1}{\beta(t)}$ . Hence, we have

$$W(t) = \frac{K(t) P(t) \beta(t)}{L(t)} \quad (24)$$

for the wage, according to (20). Equating with (23), we see that

$$\beta(t) = P(t). \quad (25)$$

For  $G$ , we have

$$G(t) = \frac{\alpha(t)}{\beta(t)} \quad (26)$$

when we solve (19) with (21) at hand. Substituting (22) and (25) into (26), we arrive at

$$\begin{aligned} G(t) &= \frac{1}{P(t)} (P^*(t) G^*(t) + (I(t) + q) (r(t) + \delta(t) - 1) - \\ & (I^*(t) + q^*) (r(t) + \delta^*(t) - 1) + q - q^*), \end{aligned} \quad (27)$$

which agrees with (12). ■

## 4. DISCUSSION

We first represent the proposed controller as a classical feedback interconnection and interpret its elements accordingly. The feedback interconnection is depicted in Fig. 1. Therein, one can see that (1) serves as a (pre)filter or as a reference whereas (3) is the plant. With this point of view, it is possible to apply the internal model principle for synchronization to interpret our results (cf. Wieland et al. [2013]). In practice, however, one would have to construct a suitable observer to reconstruct  $K^*$ ,  $I^*$ ,  $P^*$ , and  $\delta^*$ . The block containing (5), (11), (12), and (13) is both, the computation of the error and the controller itself. The signals  $K$ ,  $I$ ,  $P$ , and  $\delta$  are fed back. (22) and (23) can either be seen as a separate controller or as the inner part of a cascade, depending on where we draw our system boundaries.

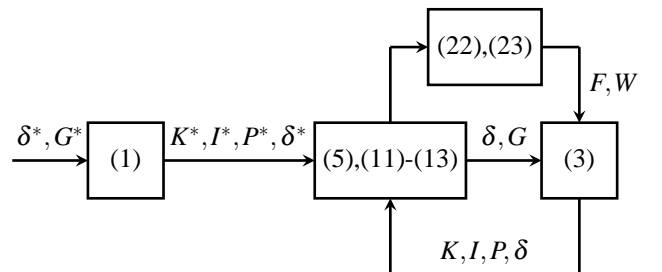


Fig. 1. Systems (1) and (3) and the feedback (5) under pinning scheme (11), (12), (13) together with  $\alpha$  and  $W$  chosen subject to formulæ (22) and (23), depicted as a classical feedback interconnection.

We now focus on the interpretation of the derived control laws themselves.

Taking a look at (11), we find that the percentage of capital stock taken as replacement investment  $\delta$  approaches 1 if the capital stock  $K$  approaches infinity, i.e. for very large capital stocks, one takes the entire capital stock as replacement investment. Moreover, as  $(\delta^* - 1)K^*$  grows,  $\delta$  grows, i.e. as the competing company increases replacement investment, one increases replacement investment as well. However, if  $\delta^*$  attains the value 1, the value vanishes. The remaining influence is a diffusive coupling known from classical synchronization problems (cf. Hale [1997]). In particular, as the difference  $I - I^*$  increases, one increases replacement investment.

Now reconsider (22) and its effect on the production quantity  $F$ . It can be inferred that an increasing product price  $P^*$  and an increasing capital stock  $K^*$  in the competing company forces one to increase the production quantity. Again, the remaining influence is diffusive, i.e. based on differences between values from (1) and (3). In particular, the diffusive term is  $\dot{I} - \dot{I}^*$  when  $PG = P^*G^* = 1$ , i.e. the difference in time-derivative of the gross investment rate between the companies affects the production quantity positively.

Last, consider (23). If larger labor input  $L$  results in the same values for capital stock  $K$  and product price  $P$ , then the wage is decreased. This can be understood such that if larger labor input does not positively influence capital stock and product price, then the labor efficiency  $\beta$  has decreased. Lower labor efficiency does thus automatically decrease wage. In contrast, if the capital stock of the company increases, wages increase as well, i.e. the employees participate in the success of their labor input.

Most of the above relations appear natural to us and it is nice to find that the constructed controller provides effects agreeing with our intuitive understanding.

Note that the above considerations are purely theoretical. While  $\delta$  may be chosen subject to (11), we do neither expect the wage to be chosen precisely according to (23), nor the product price to be chosen precisely according to (25) in practice. Instead, we aim to derive equations that support the actual choice of decision variables.

## 5. NUMERICAL EXAMPLES

To illustrate the effects of the proposed control scheme and to validate our claims, we simulate four case studies numerically.

We discuss two cases where the dynamics of (1) and (3) are decoupled, i.e. the classical case, and two cases where the dynamics of (1) and (3) are coupled according to (5), (11), (12), (13).

In all of the scenarios, we choose the initial conditions  $K_0^* = 15$  and  $I_0^* = 4$  for (1), and the initial conditions  $K_0 = 1$  and  $I_0 = 2$  for (3). The differential equations are solved in MATLAB using ode45.

Case 1 ( $\delta^* = 0, G^* = 5, \delta = 1, G = 5$ ). In the first scenario, we choose constant values for  $\delta^*, G^*, \delta$ , and  $G$ , i.e. a feedforward control. The integral curves of (1) and (3) are plotted in the upper left of Fig. 2. The company (1) is driven by a conservative strategy and constantly increases capital stock at a low rate. The start-up (3) chooses a risky strategy that provides increasing capital stock in the beginning, but then lets the capital stock go to zero asymptotically until bankruptcy.

Case 2 ( $\delta^* = 0, G^* = 5, \delta$  and  $G$  subject to (5), (11), (12), (13)). In the second scenario, we choose constant values for  $\delta^*, G^*$ , i.e. a feedforward control. The integral curves of (1) and (3) are plotted in the upper right of Fig. 2. Company (1) is driven by a conservative strategy and constantly increases capital stock at a low rate. The start-up (3) chooses the proposed control law (5), (11), (12), (13) and thus approaches the capital stock of company (1) exponentially, leading to capital stock growth

Case 3 ( $\delta^* = 0, G^* = 5, \delta = -0.5, G = 5$ ). In the third scenario, we choose constant values for  $\delta^*, G^*, \delta$ , and  $G$ , i.e. a feedforward control. The integral curves of (1) and (3) are plotted in the lower left of Fig. 2. Company (1) is driven by a conservative strategy and constantly increases capital stock at a low rate. The start-up chooses negative replacement investment (i.e. e.g. selling of machinery) and thus constantly increases capital stock until it eventually outperforms (1). Note that this scenario is not realistic as no company has infinite machinery to sell (cf. Feldstein and Rothschild [1974]). However, it illustrates that the proposed control law (5), (11), (12), (13) discussed in Case 2 may provide a performance worse than the optimum. In other words, in almost all cases it is possible to find open-loop controls that achieve better performance than the proposed control law (5), (11), (12), (13). Yet, with the proposed control law, one introduces feedback to the system and is thus capable of reacting to disturbances and uncertainties.

Case 4 ( $\delta^* = 0.3, G^* = 5, \delta$  and  $G$  subject to (5), (11), (12), (13)). In the fourth scenario, we choose constant values for  $\delta^*, G^*$ , i.e. a feedforward control. The integral curves of (1) and (3) are plotted in the lower right of Fig. 2. Company (1) is driven by a risky strategy that lets its capital stock decay asymptotically. The start-up (3) chooses the proposed control law (5), (11), (12), (13) and thus approaches the capital stock of company (1) exponentially, leading to capital stock growth in the beginning, but eventually lets capital stock decay asymptotically. This disadvantage of the proposed control strategy could be coped with by using the average of multiple companies as a reference. In this fashion, one could increase robustness against capital stock decay of single companies.

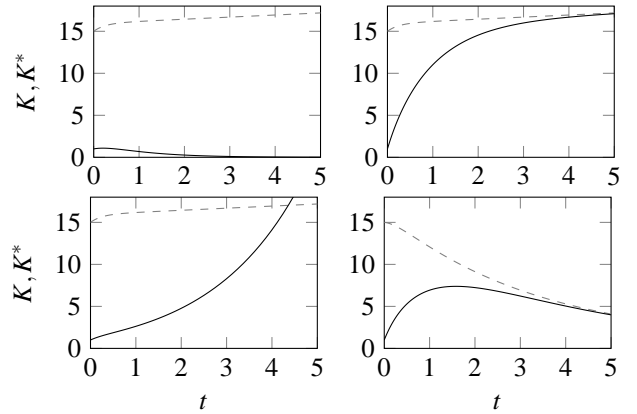


Fig. 2. Integral curves  $K(t)$  (—) and  $K^*(t)$  (- -) for systems (1) and (3) under different choices of  $\delta$  and  $G$ . Upper left:  $\delta^* = 0, G^* = 5, \delta = 1, G = 5$ . Upper right:  $\delta^* = 0, G^* = 5, \delta$  and  $G$  subject to (5), (11), (12), (13). Lower left:  $\delta^* = 0, G^* = 5, \delta = -0.5, G = 5$ . Lower right:  $\delta^* = 0.3, G^* = 5, \delta$  and  $G$  subject to (5), (11), (12), (13).

We have illustrated that our control law is not optimal in any sense. However, it introduces feedback into the system. If we

choose to apply the control law, we restrict our performance to the performance of the company (1). Even more, if the performance of the company (1) deteriorates, our performance will do so, too. In the case where the company (1) pursues a conservative strategy, our control law provided a good performance for capital growth.

## 6. A POSSIBLE EXTENSION TO COPE WITH UNCERTAINTIES

We now want to consider a setup where our estimates (or measurements) of  $K^*$ ,  $I^*$ ,  $P^*$ , and  $\delta^*$  are incorrect or somewhat imprecise. That is, we can only access the perturbed (or uncertain) signals  $K^* + \tilde{K} = K'(t)$ ,  $I^* + \tilde{I} = I'(t)$ ,  $P^* + \tilde{P} = P'(t)$ , and  $\delta^* + \tilde{\delta} = \delta'(t)$ . Therefore, Instead of (11) and (12), consider

$$\begin{aligned} C_1(K'(t), I'(t), P'(t), \delta'(t)) &= \\ \frac{1}{K(t)}(I(t) - I'(t) + \delta'(t)K'(t) - k_1(K'(t) - K(t))) & \quad (28) \\ C_2(K'(t), I'(t), P'(t), \delta'(t)) &= \\ \frac{1}{P(t)}((I(t) + q(t))(r(t) + \delta(t)) - & \\ (I'(t) + q'(t))(r(t) + \delta'(t)) + & \\ P'(t)G'(t) - k_2(I'(t) - I(t))) & \quad (29) \end{aligned}$$

with  $k_1, k_2 > 0$ . When  $k_1 = k_2 = 1$  and  $\tilde{K} = \tilde{I} = \tilde{P} = \tilde{\delta} = 0$ , this just equals (11) and (12) modulo some arithmetics. We can interpret the tuple  $(k_1, k_2)$  as a gain for the controller. If  $\tilde{K}, \tilde{I}, \tilde{P}$ , and  $\tilde{\delta}$  are not equal to zero,  $E$  does not converge to zero as time approaches infinity. However, we are able to impose an arbitrarily small positive ultimate bound on  $E$  by suitable choice of  $(k_1, k_2)$ .

In the following, let  $\mathcal{L}_\infty$  denote  $\mathcal{L}_\infty = \{f : \mathbb{R} \rightarrow \mathbb{R} | \exists c \in \mathbb{R} : |f(x)| < c \forall x \in \mathbb{R}^+\}$ .

*Claim 4.* Consider the systems (1) and (3) and the feedback (5) under pinning scheme (28), (29), (13). If  $\tilde{K}, \tilde{I}, \tilde{P}, \tilde{\delta}, \tilde{q}, K^*, I^*, P^*, \delta^*, q^*, K, I, P, \delta, q \in \mathcal{L}_\infty$ , then for every  $\xi > 0$ , there exists a tuple  $(k_1, k_2)$  such that  $E$  is ultimately exponentially bounded by  $\|E(t)\| \leq \xi$ .

**Proof.** Define the Lyapunov function candidate

$$V(E) = \frac{1}{2}E^\top E \quad (30)$$

and repeat the first steps from the proof of Claim 2 until (16). Then, substitute the feedback (5) under pinning scheme (28), (29), (13) into the Lie derivative of  $V$  along  $X$  to have

$$\begin{aligned} L_X V(E) &= (K^*(t) - K(t))(I^*(t) - \delta^*(t)K^*(t) - \\ & I'(t) + \delta'(t)K'(t) - k_1(K'(t) - K(t))) + \\ & (I^*(t) - I(t))((I^*(t) + q^*(t))(r(t) + \delta^*(t)) - P^*(t)G^*(t) - \\ & (I(t) + q(t))(r(t) + \delta(t)) + (I(t) + q(t))(r(t) + \delta(t)) - \\ & (I'(t) + q'(t))(r(t) + \delta'(t)) + P'(t)G'(t) - k_2(I'(t) - I(t))). \end{aligned} \quad (31)$$

Next, we replace  $K'$  by  $K^* + \tilde{K}$  and  $I'$  by  $I^* + \tilde{I}$ . Thereafter, we arrive at

$$\begin{aligned} L_X V(E) &= (K^*(t) - K(t))(I^*(t) - \delta^*(t)K^*(t) - \\ & I'(t) + \delta'(t)K'(t) - k_1\tilde{K}(t)) + \\ & (I^*(t) - I(t))((I^*(t) + q^*(t))(r(t) + \delta^*(t)) - P^*(t)G^*(t) - \\ & \dots \end{aligned}$$

$$\begin{aligned} \dots & (I(t) + q(t))(r(t) + \delta(t)) + (I(t) + q(t))(r(t) + \delta(t)) - \\ & (I'(t) + q'(t))(r(t) + \delta'(t)) + P'(t)G'(t) - k_2\tilde{I}(t)) + \\ & -k_1(K^*(t) - K(t))^2 - k_2(I^*(t) - I(t))^2. \end{aligned} \quad (32)$$

Now, using the  $\mathcal{L}_\infty$ -property of the signals, we can find overestimates  $\zeta_1, \zeta_2 > 0$  such that

$$L_X V(E) \leq -k_1(K^*(t) - K(t))^2 - k_2(I^*(t) - I(t))^2 + |K^*(t) - K(t)|\zeta_1 + |I^*(t) - I(t)|\zeta_2. \quad (33)$$

Introducing  $\theta_1, \theta_2 \in (0, 1)$ , we can instead write

$$\begin{aligned} L_X V(E) &\leq -k_1(1 - \theta_1)(K^*(t) - K(t))^2 - \\ & k_2(1 - \theta_2)(I^*(t) - I(t))^2 - k_1\theta_1(K^*(t) - K(t))^2 \\ & - k_2\theta_2(I^*(t) - I(t))^2 + |K^*(t) - K(t)|\zeta_1 + |I^*(t) - I(t)|\zeta_2. \end{aligned} \quad (34)$$

From the latter relation, we can see that

$$\begin{aligned} -k_1\theta_1(K^*(t) - K(t))^2 - k_2\theta_2(I^*(t) - I(t))^2 + \\ |K^*(t) - K(t)|\zeta_1 + |I^*(t) - I(t)|\zeta_2 \leq 0 \end{aligned} \quad (35)$$

implies  $L_X V(E) \leq -\gamma_3 \|E\|^\sigma$  with  $\gamma_3 = \min(k_1, k_2)$ . Condition (35) holds, if

$$\begin{aligned} |K^*(t) - K(t)|\zeta_1 &\leq k_1\theta_1(K^*(t) - K(t))^2, \\ |I^*(t) - I(t)|\zeta_2 &\leq k_2\theta_2(I^*(t) - I(t))^2 \end{aligned} \quad (36)$$

hold true. This can be simplified to

$$\begin{aligned} \frac{\zeta_1}{k_1\theta_1} &\leq |K^*(t) - K(t)|, \\ \frac{\zeta_2}{k_2\theta_2} &\leq |I^*(t) - I(t)|. \end{aligned} \quad (37)$$

If thus  $E$  exceeds the bounds provided by (37), we have  $L_X V(E) \leq -\gamma_3 \|E\|^\sigma$  with  $\gamma_3 = \min(k_1, k_2)$ . Therefore, the bounds provided by (37) form an attractive, invariant set. Moreover, the bounds can be shrunken arbitrarily by appropriate choice of the tuple  $k_1$  and  $k_2$ , which proves the assertion. ■

The existence of  $(k_1, k_2)$  for every  $\xi > 0$  is referred to as practical stability of  $E$  at the origin. In this light, the latter result bares similarities with the results obtained for practical synchronization by Montenbruck et al. [2013a,b]. Thus, we know that we can render the influence of the uncertainties arbitrarily small if we tune our gains up high enough. We want to illustrate this on an example.

Therefore, consider the setup with  $\delta^* = 0, G^* = 5$  of Cases 1-3 from previous section. We introduce bounded uncertainties of random type. In particular, we set  $\tilde{K} = 10\text{rand}$ ,  $\tilde{I} = 5\text{rand}$ ,  $\tilde{P} = 3\text{rand}$ , and  $\tilde{\delta} = 2\text{rand}$ , where  $\text{rand}$  denotes a function generating a random variable from the interval  $[-1, 1]$  at every time step. We simulate this scenario using the feedback (5) under pinning scheme (28), (29), (13) at two different gains, one of which is  $(k_1, k_2) = (0.5, 0.5)$ , and one of which is  $(k_1, k_2) = (3, 3)$ . For both cases, the integral curves of (1) and (3) are plotted in Fig. 3.

Case 1 ( $(k_1, k_2) = (0.5, 0.5)$ ). In the first scenario, we choose control gains lower than the nominal gains (1, 1). The integral curves of (1) and (3) are plotted in the left of Fig. 3. The company (1) is driven by a conservative strategy and constantly increases capital stock at a low rate. The start-up (3) receives uncertain measurements of the states of (1) but is capable of increasing capital stock and converging into an  $\xi$ -neighborhood

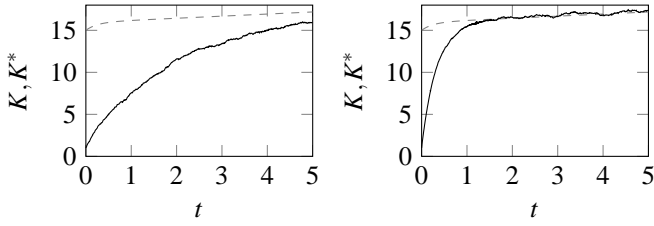


Fig. 3. Integral curves  $K(t)$  (—) and  $K^*(t)$  (- - -) for systems (1) and (3) subject to the feedback (5) under pinning scheme (28), (29), (13) for different choices of control gains  $(k_1, k_2)$  and random disturbances  $\tilde{K}, \tilde{I}, \tilde{P}, \tilde{\delta} \in \mathcal{L}_\infty$ . Left:  $(k_1, k_2) = (0.5, 0.5)$ . Right:  $(k_1, k_2) = (3, 3)$ .

of  $K^*$ , that appears to be large when compared to the magnitude of capital stock.

Case 2 ( $(k_1, k_2) = (3, 3)$ ). In the second scenario, we choose control gains higher than the nominal gains  $(1, 1)$ . The integral curves of (1) and (3) are plotted in the right of Fig. 3. The company (1) is driven by a conservative strategy and constantly increases capital stock at a low rate. The start-up (3) receives uncertain measurements of the states of (1) but is capable of increasing capital stock and converging into an  $\xi$ -neighborhood of  $K^*$ , that appears to be small when compared to the magnitude of capital stock.

We have illustrated that the proposed control scheme is, by construction, able to cope with uncertainties. Even in presence of inexact measurements of the states of (1), our company may practically stabilize the error  $E$  at the origin. That is, if the uncertainties and the states remain bounded (in the  $\mathcal{L}_\infty$ -sense), for any arbitrarily small but positive upper bound  $\xi$ , we can choose control gains  $(k_1, k_2)$  such that the error  $E$  is ultimately bounded by  $\xi$ . In the simulated scenarios, we were able to cope with relatively large uncertainties at comparatively low gains.

## 7. CONCLUSIONS

Using the the model of a rationally managed firm in a competitive industry proposed by Gould [1968], we set up a scenario of two competing companies, one of which was assumed to be well-established, well-running and one of which was assumed to be a start-up, whose point of view we took. Instead of using the model to derive optimal (possibly open-loop) inputs, we decided to construct controllers that force the capital stock and gross investment rate of our company to converge to the capital stock and gross investment rate of the other company, therefore letting the other company define our desired performance. By exploiting the QUAD property of the vector field of the underlying model, we were able to construct a pinning controller that exponentially stabilized the difference between the companies at the origin. Thereafter, we showed how to choose production quantity and wage such that our control law can be realized. We consequently illustrated possible advantages and disadvantages of the proposed approach in numerical examples. It turned out that the proposed control law is disadvantageous if the capital stock of the well-established company decays. This may be coped with by using the average of multiple companies as a reference. Last, we described how it is possible to cope with uncertainties in the described setting, attesting the practicality of the control law.

## REFERENCES

- N. Basalto, R. Bellotti, F. De Carlo, P. Facchi, and S. Pascazio. Clustering stock market companies via chaotic map synchronization. *Physica A*, 345:196–206, 2005.
- L. Chen and G. Chen. Controlling chaos in an economic model. *Physica A*, 374:349–358, 2007.
- T. Chen, X. Liu, and W. Lu. Pinning complex networks by a single controller. *IEEE Trans. Circuits Syst. I*, 54:1317–1326, 2007.
- P. DeLellis, M. di Bernardo, and F. Garofalo. Synchronization of complex networks through local adaptive coupling. *Chaos*, 18:037110, 2008.
- P. DeLellis, M. di Bernardo, and F. Garofalo. Novel decentralized adaptive strategies for the synchronization of complex networks. *Automatica*, 45:1312–1318, 2009.
- P. DeLellis, M. di Bernardo, and G. Russo. On QUAD, Lipschitz, and contracting vector fields for consensus and synchronization of networks. *IEEE Trans. Circuits Syst. I*, 58:576–583, 2011.
- M. S. Feldstein and M. Rothschild. Towards an economic theory of replacement investment. *Econometrica*, 42:393–424, 1974.
- J.P. Gould. Adjustment costs in the theory of investment of the firm. *Rev. Econ. Stud.*, 35:47–55, 1968.
- R.O. Grigoriev, M.C. Cross, and H.G. Schuster. Pinning control of spatiotemporal chaos. *Phys. Rev. Lett.*, 79:2795–2798, 1997.
- H. Hakansson and D. Ford. How should companies interact in business networks? *J. Bus. Res.*, 55:133–139, 2002.
- J. K. Hale. Diffusive coupling, dissipation, and synchronization. *J. Dyn. Differ. Equ.*, 9:1–52, 1997.
- L. S. Karp and J. M. Perloff. Open-loop and feedback models of dynamic oligopoly. *Int. J. Ind. Organ.*, 11:369–389, 1993.
- H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 1996.
- X. Li, X. Wang, and G. Chen. Pinning a complex dynamical network to its equilibrium. *IEEE Trans. Circuits Syst. I*, 51:2074–2087, 2004.
- J. M. Montenbruck, M. Bürger, and F. Allgöwer. Practical cluster synchronization of heterogeneous systems on graphs with acyclic topology. In *Proc. 52nd IEEE Conf. Decision and Control*, 2013a.
- J. M. Montenbruck, G. S. Seyboth, and F. Allgöwer. Practical and robust synchronization of systems with additive linear uncertainties. In *Proc. 9th IFAC Symp. Nonlinear Control Systems*, pages 743–748, 2013b.
- J. D. W. Morecroft. The feedback view of business policy and strategy. *Syst. Dynam. Rev.*, 1:4–19, 1985.
- J. D. W. Morecroft. *Strategic Modelling and Business Dynamics: A Feedback Systems Approach*. Wiley, 2007.
- M. Porfiri and M. di Bernardo. Criteria for global pinning-controllability of complex networks. *Automatica*, 44:3100–3106, 2008.
- M. A. Simaan and T. Takayama. Optimum monopolist control in a dynamic market. *IEEE Trans. Syst. Man Cyber.*, 6:799–807, 1976.
- W. Souma, Y. Fujiwara, and H. Aoyama. Complex networks and economics. *Physica A*, 324:396–401, 2003.
- J. Sterman. *Business Dynamics*. McGraw-Hill, 2000.
- S. H. Strogatz. Exploring complex networks. *Nature*, 410:268–276, 2001.
- P. Wieland, J. Wu, and F. Allgöwer. On synchronous steady states and internal models of diffusively coupled systems. *IEEE Trans. Automat. Contr.*, 58:2591–2602, 2013.