Practical Cluster Synchronization of Heterogeneous Systems on Graphs with Acyclic Topology*

Jan Maximilian Montenbruck, Mathias Bürger, and Frank Allgöwer

***Abstract*— We study the problem of practical cluster synchronization in networks of heterogeneous dynamical systems. The considered framework involves groups of identical dynamical systems, interacting with each other through linear couplings. The control objective studied in this paper is to achieve synchronization up to a possibly small error of all identical systems. Based on the two assumptions that all systems satisfy the QUAD condition and that the global coupling structure is acyclic, a constructive procedure for a coupling design ensuring practical cluster synchronization is proposed. For establishing the desired result, first, the synchronization of identical systems under external disturbances is studied. The main contribution is the extension of this result to the complete heterogeneous network. The theoretical results are illustrated and tested numerically on an exemplary network composed of several Van der Pol oscillators and Chua's circuits.**

I. INTRODUCTION

Many real world networks are composed of systems with different dynamical behavior. The nature of these networks is *heterogeneous*. In general, synchronization, which is convergence of all solutions to a common trajectory, is hard to achieve in such heterogeneous settings. In fact, diffusive couplings often not suffice to synchronize such systems. Instead, local controllers such as pinning controllers are employed to force the systems to synchronize. However, one can observe approximate synchronization in physical systems when solely using diffusive couplings. Therefore, we will herein analyze the dynamics of a diffusively coupled network of heterogeneous systems composed of networks of identical subsystems. In this setting, we will derive sufficient conditions for the network to synchronize approximately. In particular, for an arbitrary upper bound for the synchronization error, we will be able to derive a gain such that the synchronization error can be overestimated by this bound, thus motivating practical synchronizability.

Related Work. The phenomenon of synchronization has been widely studied in the engineering and physics community and can be traced back to Mirollo and Strogatz [1] or Pecora and Caroll [2]. Over the years, several fields of research have emerged from this. Among them, there is the synchronization of heterogeneous (nonidentical) systems, cluster synchronization, and pinning control. Naturally, synchronization of heterogeneous systems is more complex than in homogeneous systems and both, cluster synchronization and pinning control can be studied on either of them. Cluster synchronization refers to the emergence of different synchronized solutions of *clusters* of systems. It has been studied in heterogeneous second-order nonlinear systems under diffusive coupling [3]–[5], in heterogeneous linear and homogeneous nonlinear systems under diffusive coupling [6], in heterogeneous nonlinear systems with two clusters assuming identical inputs for every cluster [7], in homogeneous networks under pinning control [8], and in nonlinear heterogeneous systems under pinning control in community networks [9]. Heterogeneous synchronization without clustering has been applied in community networks under adaptive coupling strength [10], using pinning control [11], and using the internal-model principle [12]–[15]. Robust synchronization has been considered using techniques similar to the ones proposed herein [16].

An important concept in synchronization studies is the QUADcondition. In fact, there have been significant contributions showing that the QUAD condition is a key property facilitating synchronization [17]–[20]. The QUAD condition becomes particularly relevant as it opens a way for synchronization proofs based on Lyapunov theory. We build in this paper upon the previous results on QUAD systems, and exploit the Lyapunov structure of the synchronization proofs.

Contributions. We propose a solution to the problem of practical cluster synchronization in networks composed of different classes of dynamical systems. We consider a network structure, where each node represents a dynamical system. Nodes governed by the same differential equations are said to be a class of systems. We consider a fixed interaction structure between the nodes and show how under certain assumptions, an adjustment of coupling gains between identical systems suffices to achieve cluster synchronization up to a small error. As a first contribution, we show that a network of identical systems satisfying the QUAD condition can be practically synchronized by a linear coupling, even in the presence of external disturbances. We show therefore that the local coupling gains influence the ultimate synchronization error in a reciprocal manner. As a main contribution, we show how this result can be used to achieve practical cluster synchronization in networks with acyclic topology. We prove that, given a certain set of allowed bounds for the synchronization errors, one can always choose the local coupling gains such that the synchronization errors are all ultimately bounded by the desired bounds. The proof of this result is constructive and provides an explicit algorithmic procedure for the design of the coupling gains. Finally, we present an

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exemplary network consisting of Van der Pol oscillators and Chua's circuits supporting the theoretical findings.

Organization of the Paper. The paper is structured as follows. The problem setup considered in the paper is introduced in Section II. In Section II.A the considered class of dynamic systems is introduced. In Section II.B, the considered coupling structure between these systems is discussed. Section III provides the first theoretical results, giving a bound on the synchronization errors. First, in Section III.A the stability of the errors for bounded inputs is proven, then, in Section III.B, a result is presented which shows that arbitrary small error bounds can be achieved by choosing sufficiently large gains. The main theorem is contained in Section IV, where the prior results are extended to the complete heterogeneous network, providing a constructive result for practical cluster synchronization. The paper concludes with a numerical example in Section V and final remarks in Section VII.

Notation. In the remainder of this paper, variables are formatted italic, spaces are double struck and operators are written upright, whereas calligraphic letters represent graphs. In particular, $\mathbb{R}$ is the field of real numbers. If the negative or positive numbers should explicitly be excluded, $\mathbb{R}^+$ and $\mathbb{R}^-$ is used, respectively. The inclusion or exclusion of $\{0\}$ will explicitly be noted in every case through $\cup\{0\}$ or $\setminus\{0\}$, respectively. The n-times Cartesian product on the real field $\mathbb{R}\times\cdots\times\mathbb{R}$ (the space of n-tuples) is abbreviated by $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ are the matrices composed of m n-tuples; spec is the spectrum, max the maximum, min the minimum, sup the supremum and inf the infimum; $\top$ denotes the transpose and $\|\cdot\|$ is the Euclidean norm or the induced Euclidean norm, respectively. A $[0,\infty)\to[0,\infty)$ function is said to be class $\mathscr{K}$, if it is zero at zero, strictly increasing, and continuous, and a $[0,\infty)^2\to[0,\infty)$ function is class $\mathscr{KL}$, if it is class $\mathscr{K}$ in the first argument and decreasing to zero in the second argument.

II. PROBLEM STATEMENT

We consider networks of heterogeneous nonlinear dynamical systems influencing each other in a linear manner. The control objective is to achieve practical synchronization, i.e., synchronization up to a pre-specified, possibly small error, of all systems governed by the same dynamics.

A. Systems

Networks consisting of N dynamical systems are considered, where each system is described by one of M $(M\leq N)$ different nonlinear differential equations. For each class of dynamics $i\in\{1,\dots,M\}$, there are N_i systems governed by this dynamics. We allow in particular $N_i=1$, and note that $\sum_{i=1}^{M}N_i=N$. The state vector of each dynamical system Σ_{ij}, $j\in\{1,\dots,N_i\}, i\in\{1,\dots,M\}$ is denoted by $x_{ij}\in\mathbb{R}^{n_i}$ and evolves according to the dynamics

$$\dot{x}_{ij}=f_i(x_{ij})+u_{ij}. \tag{1}$$

Here $u_{ij}\in\mathbb{R}^{n_i}$ and is the input of the ijth system, through which the coupling with neighboring systems will be realized. The main assumption we impose on the systems is that all nonlinear functions f_i are QUAD .

Definition 1 ([19]): A vector field $f(x)$ is said to be QUAD (Δ,ω), if there exists some $\omega\in\mathbb{R}^+\setminus\{0\}$, $\Delta=[\Delta_{ij}]$, $\Delta_{ij}=0$ for all $i\neq j$, such that the quadratic inequality $(x_a-x_b)^\top(f(x_a)-f(x_b)) - (x_a-x_b)^\top\Delta(x_a-x_b) \leq -\omega(x_a-x_b)^\top(x_a-x_b)$ holds for all x_a,x_b.

Considering QUAD systems is clearly a restrictions. However, QUAD systems have been proven to be of significant importance in the synchronization literature [17]–[20]. This justifies to focus on systems having this desirable property. The average of the states of all systems within the same class $i\in\{1,\dots,M\}$ is in the following denoted by

$$s_i(t)=\frac{1}{N_i}\sum_{j=1}^{N_j}x_{ij}(t).$$

The deviation from this trajectory will be called the synchronization error $e_{ij}(t)=x_{ij}(t)-s_i(t)$. Please note that this definition of the synchronization error implies $\sum_{j=1}^{N_i}e_{ij}=0$, where 0 denotes the all zeros vector.

Now, some notions regarding the collection of systems are required. The *class* of systems equipped with identical differential equations is denoted as $\Sigma_i=\bigcup_{j=1}^{N_i}\Sigma_{ij}$ and the collection of all classes is $\Sigma=\bigcup_{i=1}^{M}\Sigma_i$. If one class k should explicitly be excluded, the terminology $\Sigma_{\setminus k}=\bigcup_{j\neq k}^{M}\Sigma_i=\Sigma\setminus\Sigma_k$ is used. If the classes with a higher index than k are considered, $\Sigma_{>k}$ abbreviates $\Sigma_{>k}=\bigcup_{j=k+1}^{M}\Sigma_j$.

Definition 2: A collection of systems classes Σ is said to be homogeneous if $M=1$, and heterogeneous if $M>1$.

In the same way, the states of systems, system classes, and collections of system classes can be defined. The vector x_i denotes the stacked vector of all x_{ij}, i.e., $x_i^\top=\left[x_{i1}^\top\cdots x_{iN_i}^\top\right]$, and x the stacked vector of all x_i, i.e, $x^\top=\left[x_1^\top\cdots x_M^\top\right]$. Furthermore, $x_{\setminus k}$ explicitly excludes the states from x_k, and $x_{>k}$ are the states of classes with a higher index than k. We take the analog notation for inputs u_i, u, errors e_i, e, and functions $f^\top(x)=\left[f_i^\top(x_{i1})\cdots f_M^\top(x_{MN_M})\right]$. Then the dynamical representation of all system is given in a compact form by $\dot{x}=f(x)+u$.

B. Couplings

In general, *global* couplings are considered in this paper. Global couplings are couplings between systems in different classes. However, for clarity of presentation, the description of these couplings is approached by first introducing *local* couplings, which are couplings among members of one class.

Let the systems Σ_i be placed at the vertices of a directed, weighted graph $\mathscr{G}_i$, represented through the matrix $W_i\in\mathbb{R}^{N_i\times N_i}$ containing its edge weights. The graph $\mathscr{G}_i$ will in the following also be called a *local* graph. An element $W_{i,mn}$ of this weight matrix is a scalar that describes whether or not the output of system n is used as input for system m and how much the function is scaled in between; if there is no edge from n to m, then $W_{i,mn}=0$. In addition, let every system admit an external input $\tilde{u}_{ij}$, which will later describe the influence of systems in other classes $\Sigma_{\setminus i}$. Now, the input to one system can be written as $u_{ij}=\tilde{u}_{ij}+\sum_{n=1}^{N_i}W_{i,jn}x_{in}$. Taking into account all systems on the subgraph $\mathscr{G}_i$, their

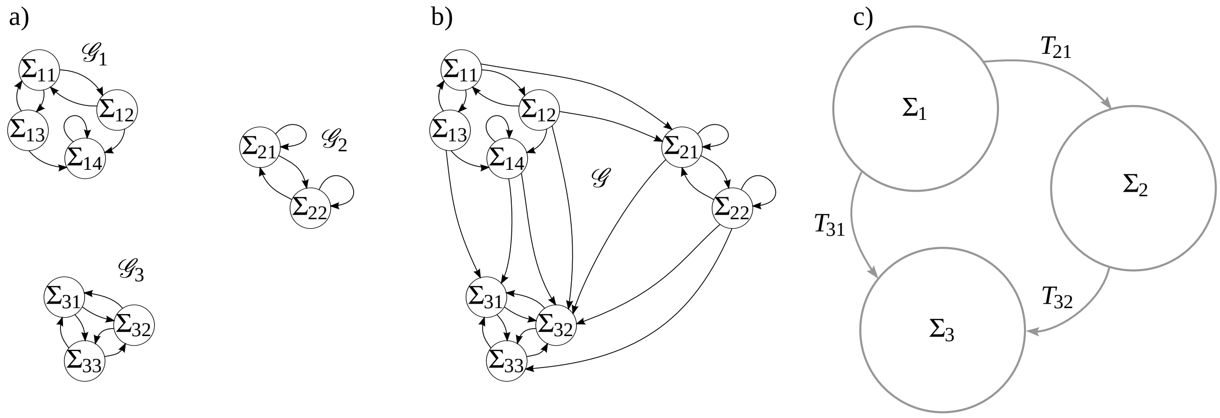

Fig. 1. Structure of the underlying graphs: a) A class is *locally* connected through a graph $\mathscr{G}_i$ b) The graphs $\mathscr{G}_i$ are subgraphs of a *global* graph $\mathscr{G}$ that also contains the couplings between different classes c) The topology of $\mathscr{G}$ is represented by a graph $\mathscr{T}$ that is assumed to be acyclic.

couplings and external influences can be written in vector form as $u_i = \tilde{u}_i + (W_i \otimes I_n)x_i$.

There shall be little restriction on the structure of $\mathscr{G}_i$ (particularly including admission of self-loops), but certain assumptions on the structure of the coupling matrices W_i are required. Let $\mathbb{W}^n$ denote the set $\mathbb{W}^n = \{A \in \mathbb{R}^{n\times n} | A1_n = 0, \operatorname{rank}(A) = n-1, A = A^\top, \max(\operatorname{spec}(A) \setminus \{0\}) < 0\}$, where $1_n = [1 \cdots 1]^\top \in \mathbb{R}^n$, and $\mathbb{W} = \bigcup_{n=2}^{\infty} \mathbb{W}^n$. Note that $W_i \in \mathbb{W}$ means that W_i is the negative Laplacian of some connected, balanced graph [21] and we denote

$$\max(\operatorname{spec}(W_i) \setminus \{0\}) = \lambda_i, \tag{2}$$

The conceptual idea we exploit to achieve synchronization of all systems within Σ_i is as follows. We introduce one common positive gain $g_i \in \mathbb{R}^+ \setminus \{0\}$ which is used by all systems Σ_i to amplify the influence from other systems in the same class. That is, the coupling matrix W_i is multiplied with the gain g_i. For convenience, we denote the novel, scaled coupling matrix as $W_{ii} = W_i g_i$. The control input to the systems in Σ_i is now given by

$$u_i = \tilde{u}_i + (W_{ii} \otimes I_n)x_i. \tag{3}$$

We are now ready to consider the connection between systems in different classes. On the larger scale, a second graph $\mathscr{G}$ is introduced. This graph has the local graphs $\mathscr{G}_i$ as proper subgraphs. Additionally, edges connecting systems from different classes are introduced. We call $\mathscr{G}$ the *global* graph, and describe the adjacency relation between the nodes by the matrix $W \in \mathbb{R}^{\sum_{i=1}^M N_i \times \sum_{i=1}^M N_i}$. In fact, the matrix W can be written as a block matrix with matrix entries $W_{ij} \in \mathbb{R}^{N_i \times N_j}$ that describe whether or not the outputs of class Σ_j are used as inputs for class Σ_i and how much these functions are scaled in between. Naturally, the diagonal matrices of W are the mappings from the outputs of Σ_i to itself, given by W_{ii}, and the external inputs $\tilde{u}_i$ are described by the off-diagonal elements of W through $\tilde{u}_i = \sum_{j\neq i}^M (W_{ij} \otimes I_n)x_j$. It follows that an element $W_{ij,mn}$ scales the output of system Σ_{jn} as input of system Σ_{im}. The dimensions of all couplings result from above considerations, as $W \in \mathbb{R}^{\sum_{i=1}^M N_i \times \sum_{i=1}^M N_i}$, $W_i \in \mathbb{R}^{N_i \times N_i}$, $W_{ij} \in \mathbb{R}^{N_i \times N_j}$, and $W_{i,mn}, W_{ij,mn} \in \mathbb{R}$. This large-scale point of view leads to the complete definition of the input signals to one system as

$$u_{ij} = \sum_{n=1}^{N_1} W_{i1,jn} x_{1n} + \cdots + \sum_{n=1}^{N_M} W_{iM,jn} x_{Mn} = \sum_{m=1}^{M} \sum_{n=1}^{N_i} W_{im,jn} x_{mn}. \tag{4}$$

As a consequence, the vector u_i is given by $u_i = \sum_{j=1}^M (W_{ij} \otimes I_n)x_j$, and the vector u simplifies to $u = (W \otimes I_n)x$ such that the dynamics can be written as

$$\dot{x} = f(x) + (W \otimes I_n)x. \tag{5}$$

For the purpose of this paper, we need to impose restrictions on the interconnections between systems in different classes. In a first step, we simplify the overall topology of $\mathscr{G}$ to a novel abstract (unweighted, directed) graph $\mathscr{T}$. The adjacency matrix of $\mathscr{T}$ is denoted $T \in \mathbb{R}^{M\times M}$ and its elements are defined as

$$T_{ij} = \begin{cases} 0 & \text{if } W_{ij,mn} = 0 \quad \forall\, m = 1, \cdots, N_i, n = 1, \cdots, N_j, \\ 1 & \text{else.} \end{cases}$$

A zero element T_{ij} indicates that the matrix W_{ij} is the all zeros matrix. Thus, $\mathscr{T}$ simplifies the microscopic view on scalar coupling weights to a macroscopic view on zeros (there is no coupling between classes i and j) and ones (there is some coupling between classes i and j). The main assumption we impose on the global coupling structure is that the graph $\mathscr{T}$ is *acyclic*. A directed graph is said to be acyclic if it has no path starting and ending at one vertex.

Definition 3 ([22]): A directed graph $\mathscr{T}$ is said to be acyclic, if it contains no closed paths.

A main result, important for the purpose of this paper, is the following. Every acyclic directed graph has a permutation such that its adjacency matrix becomes triangular [22]. A graph with triangular adjacency matrix is called a topologically ordered acyclic directed graph.

Definition 4 ([22]): A directed graph $\mathscr{T}$ with adjacency matrix $T = [T_{ij}]$ is a topologically ordered acyclic directed graph, if $T_{ij} = 0\ \forall i > j$.

Tere are powerful algorithms to find such permutations [22]. Thus, we assume in the following that $T_{ij} = 0\ \forall i > j$. In other words, a necessary condition for $\mathscr{T}$ to be topologically ordered is that, if there is some output from a member of class Σ_j used as input for some member of class Σ_i, then there shall be no connection vice versa.

An exemplary setting, involving the different notions of graphs considered here, is depicted in Fig. 1. In the depicted setting, the local graphs $\mathscr{G}_1$, $\mathscr{G}_2$, and $\mathscr{G}_3$ are shown in Fig. 1 a). Then, in Fig. 1 b), the edges between systems from different classes are added, leading to the global graph $\mathscr{G}$. Notably, there is no path starting in a class i and leaving class i that can end in class i for all $i = 1,2,3$. Thus, the topology $\mathscr{T}$ of $\mathscr{G}$ is shown in c) and turns out to be acylic. It is also topologically ordered as $T_{21} = T_{32} = T_{31} = 1$ and $T_{12} = T_{23} = T_{13} = 0$.

III. Practical Synchronization

The main objective of this paper is to establish a practical cluster synchronization of the complete network. That is, we want to ensure that all systems within one class synchronize up to an error that can be chosen a-priori. To achieve this objective, we focus first on the synchronization problem for systems within one class and consider the influence of external input signals. In fact, we show that if the coupling gains are chosen sufficiently large, synchronization with an arbitrarily small error is achieved.

A. Input to Error Stability

In the first place, we assume that $\tilde{u}_i$ is only known to be bounded but is unknown otherwise. The main finding of this section is that for sufficiently large g_i, a bounded $\tilde{u}_i$ results in a bounded e_i. Furthermore, the bound on $\tilde{u}_i$ and the gain g_i determine the ultimate bound on e_i.

Definition 5: A class of systems Σ_i is said to be input to error stable, if the error e_i is bounded by

$$\|e_i(t)\| \leq \beta_i(\|e_i(0)\|,t) + \varepsilon_i \sup_{0\leq\tau\leq t} \|\tilde{u}_i(\tau)\|,$$

where β_i is some class $\mathscr{KL}$ function and ε_i a finite gain. We are now ready to introduce the main result of this section.

Theorem 1: Consider a class Σ_i of systems (1) with coupling (3), assume $W_i \in \mathbb{W}$, and let (2) hold. Furthermore, let f_i be QUAD (Δ_i, ω_i) with some c_i satisfying $\Delta_i - \omega I_{n_i} \leq c_i I_{n_i}$. Then, for any g_i chosen such that $c_i + g_i\lambda_i < 0$, Σ_i is input to error stable with $\varepsilon_i = \frac{1}{-(c_i+g_i\lambda_i)}$.

Proof: Consider the Lyapunov function candidate $V_i = \frac{1}{2}\sum_{j=1}^{N_i} e_{ij}^\top e_{ij}$. The directional derivative takes the form $\dot{V}_i = \sum_{j=1}^{N_i} e_{ij}^\top(\dot{x}_{ij} - \dot{s}_i)$. The sum $\sum_{j=1}^{N_i} e_{ij}^\top \dot{s}_i$ is equal to zero since $\sum_{j=1}^{N_i} e_{ij} = 0$. Similarly,$\sum_{j=1}^{N_i} e_{ij}^\top f_i(s_i) = 0$. Hence

$$\dot{V}_i = \sum_{j=1}^{N_i} e_{ij}^\top \left(f_i(x_{ij}) - f_i(s_i) + \sum_{m=1}^{M}\sum_{n=1}^{N_i} W_{im,jn} x_{mn} \right).$$

Now, $\sum_{m=1}^{M}\sum_{n=1}^{N_i} W_{im,jn} x_{mn}$ can be partitioned into $\tilde{u}_{ij}$ and $\sum_{n=1}^{N_i} g_i W_{i,jn} x_{in}$. We can now simply subtract $\sum_{n=1}^{N_i} g_i W_{i,jn} s_i = 0$, which is zero due to $W_{ii} \in \mathbb{W}$, to arrive at

$$\dot{V}_i = \sum_{j=1}^{N_i} e_{ij}^\top \left(f_i(x_{ij}) - f_i(s_i) + \sum_{n=1}^{N_i} g_i W_{i,jn} e_{in} + \tilde{u}_{ij} \right).$$

The next step is to use the QUAD condition, which provides, together with the upper bound c_i [19], the bound

$$\dot{V}_i \leq \sum_{j=1}^{N_i} e_{ij}^\top \left(c_i e_{ij} + \sum_{n=1}^{N_i} g_i W_{i,jn} e_{in} + \tilde{u}_{ij} \right).$$

This bound can be rewritten in vector form as $\dot{V}_i \leq c_i e_i^\top e_i + e_i^\top (W_{ii} \otimes I_n) e_i + e_i^\top \tilde{u}_i$. From the definition of the synchronization error, $\sum_{j=1}^{N_i} e_{ij} = 0$. By our assumption on the local coupling structure, i.e., $W_{ii} \in \mathbb{W}$, we can directly obtain the bound $e_i^\top (W_{ii} \otimes I_n) e_i \leq g_i \lambda_i e_i^\top e_i$, where λ_i is as defined in (2). Further, using $e_i^\top \tilde{u}_i \leq \|e_i\| \|u_i\|$, the bound on the directional derivative can be refined as

$$\begin{aligned} \dot{V}_i &\leq (c_i + g_i\lambda_i) e_i^\top e_i + \|e_i\| \|\tilde{u}_i\| \\ &= (c_i + g_i\lambda_i)(1-\theta) e_i^\top e_i + (c_i + g_i\lambda_i)\theta e_i^\top e_i + \|e_i\| \|\tilde{u}_i\| \end{aligned} \tag{6}$$

for some $0 < \theta < 1$. Now, the gain g_i can always be chosen large enough such that $c_i + g_i\lambda_i < 0$. The bound (6) can now be equivalently written as

$$\dot{V}_i \leq (c_i + g_i\lambda_i)(1-\theta) e_i^\top e_i \quad \forall\ \|e_i\| \geq \frac{1}{-\theta(c_i + g_i\lambda_i)} \|u_i\|.$$

In fact, this implies that $\dot{V}_i \leq 0$ if $\|e_i\| \geq \frac{1}{-\theta(c_i+g_i\lambda_i)} \|u_i\|$. A standard argumentation, as e.g. used in the proof of Theorem 4.18 in [23], can now be applied to show that Σ_i is input to error stable with $\varepsilon_i = \frac{1}{-(c_i+g_i\lambda_i)}$. ■

The key aspects of this result are, first, that the coupling gains g_i can always be chosen sufficiently large such that synchronization with a bounded error is achieved, and, second, the ultimate synchronization error depends reciprocal on the gains. Thus, the local coupling gains can be used to reduce the influence of external perturbations on the synchronization.

B. Practical Synchronizability

For the next step, we assume that the bound on the signal $\tilde{u}_i$ is known as $\sup_{0\leq\tau\leq t} \|\tilde{u}_i(\tau)\| = z_i$.

Definition 6: A class of systems Σ_i is said to be practically synchronizable, if, for any choice of $\varepsilon_i' > 0$, there exists a coupling gain g_i, such that the error e_i is bounded by

$$\|e_i(t)\| \leq \beta_i(\|e_i(0)\|,t) + \varepsilon_i',$$

where β_i is some class $\mathscr{KL}$ function.

Practical synchronizability can now be readily established for the considered class of systems.

Theorem 2: Consider a class Σ_i of systems (1) with coupling (3), assume $W_i \in \mathbb{W}$, and let (2) hold. Furthermore, let f_i be QUAD (Δ_i, ω_i) with some c_i satisfying $\Delta_i - \omega I_{n_i} \leq c_i I_{n_i}$ and let $z_i = \sup_{0\leq\tau\leq t} \|\tilde{u}_i(\tau)\|$ be finite and known. Then Σ_i is practically synchronizable.

Proof: Using Theorem 1 and the bound z_i for $\|\tilde{u}_i(\tau)\|$, one can directly conclude that

$$\|e_i(t)\| \leq \beta_i(\|e_i(0)\|,t) + \frac{1}{-(c_i + g_i\lambda_i)} z_i. \tag{7}$$

Now, given any ε_i', the gain g_i can always be chosen sufficiently large such that

$$g_i \geq \frac{z_i + \varepsilon_i' c_i}{-\varepsilon_i' \lambda_i}. \tag{8}$$

Substituting (8) in (7), $\|e_i(t)\| \leq \beta_i(\|e_i(0)\|, t) + \varepsilon_i'$ follows, which corresponds to Definition 6. ■

Note that the bound ε_i' has been chosen a-priori and can, in particular, be chosen arbitrarily small. Thus, in the limit, i.e., as g_i approaches infinity, exact synchronization can be achieved. However, for finite gains a synchronization error will always remain.

IV. Practical Cluster Synchronization

So far, only local couplings have been considered, as depicted in Fig. 1 a). Now, the discussion is extended to include couplings between different system classes, as illustrated in Fig. 1 b). Thus, coupling (4) is now considered instead of (3) and $\tilde{u}_i$ is no longer anonymous. However, in the general setup involving heterogeneous systems, exact synchronization cannot be achieved. Thus, an alternative notion of synchronization is required. The concept we employ here is cluster synchronization. That is, we want all systems within one system class to synchronize. Unfortunately, due to the persistent influence from systems in other classes, an exact cluster synchronization cannot be achieved. Therefore, we refine the discussion to the concept of practical cluster synchronization, where systems within one class synchronize up to a pre-specified error. As a conceptual idea, we aim to design the local coupling gains, used by all systems within one class, to achieve such a practical cluster synchronization.

Definition 7: A collection Σ of classes Σ_i is said to be practically cluster synchronizable, if, for any choice of bounds $\{\varepsilon_1', \cdots \varepsilon_M'\}$, there exists a set of gains $\{g_1, \cdots g_M\}$, such that

$$\|e_i(t)\| \leq \beta_i(\|e_i(0)\|, t) + \varepsilon_i'$$

holds $\forall\ i = 1, \cdots, M$, where β_i is some class $\mathscr{KL}$ function.

The line of argumentation we employ here can be sketched as follows. The result of Theorem 1 shows that g_i directly influences the ultimate bound on e_i. Thus, knowing z_i gives rise to a situation where the ultimate bound on e_i can be made arbitrarily small with sufficiently high gains. Now, considering the global coupling structure it follows directly that the "external" input $\tilde{u}_i$ is fully defined by the states of other systems, i.e., $z_i = \sup_{0\leq\tau\leq t}\|\tilde{u}_i(\tau)\| = H\left(\sup_{0\leq\tau\leq t}\|x_{\setminus i}(\tau)\|\right)$, where H is some (linear) function of W. By exploiting this observation, we can establish practical cluster synchronizability under the assumption that the graph $\mathscr{T}$ is acyclic.

Theorem 3: Consider a collection Σ of classes Σ_i of systems (1) with coupling (4). Assume that $\mathscr{T}$ is acyclic, $\forall i = 1, \cdots, M$: $W_i \in \mathbb{W}$ and (2) holds. Furthermore, let f_i be QUAD(Δ_i, ω_i) with some c_i satisfying $\Delta_i - \omega I_{n_i} \leq c_i I_{n_i}$. Then Σ is practically cluster synchronizable.

Proof: The proof follows from induction. First, recall that there exists a permutation for T, such that $\mathscr{T}$ becomes a topologically ordered acyclic directed graph [22]. Hence $T_{ij} = 0\ \forall\ i > j$. Thus there is at least one class Σ_M, for that $\tilde{u}_M = 0$ in (3) holds. This class will be the basis for the following induction.

Basis ($i = M$): At the basis, due to $\mathscr{T}$ being topologically ordered, $T_{Mj} = 0$ for $j \neq M$. Thus, $\mathscr{W}_{Mj} = 0$ for $j \neq M$ and hence $z_M = 0$. Then, $\lim_{t\to\infty} e_M = 0$ for g_M satisfying $g_M \geq \frac{c_M}{-\lambda_M}$, which is (8) with $z_M = 0$.

Hypothesis ($1 < i < M$): Σ_i is practically synchronizable with $\|e_i(t)\| \leq \beta_i(\|e_i(0)\|, t) + \varepsilon_i'$.

Step ($i-1$): Recalling the form of (5) and the topological ordering of $\mathscr{T}$, then Σ_{i-1} only receives input from $\Sigma_{>i}$ and $z_{i-1} = H_{i-1}\sup_{0\leq\tau\leq t}\|x_{>i-1}\|$, where $H_{i-1} = [W_{ii}\otimes I_n \cdots W_{iM}\otimes I_n]$. Following the hypothesis, $\|e_i(t)\|$ goes to ε_i'. Thus, using the triangle inequality, $\|x_{>i-1}\|$ can be estimated via $\|x_{>i-1}\| \leq \|s_{>i-1}\| + \sqrt{\sum_{j=i}^M \varepsilon_j'^2}$. In doing so, z_{i-1} is known and g_{i-1} can be set such that $g_{i-1} \geq \frac{z_{i-1}+\varepsilon_{i-1}'c_{i-1}}{-\varepsilon_{i-1}'\lambda_{i-1}}$. This is just Theorem 2, which makes Σ_{i-1} satisfy $\|e_{i-1}(t)\| \leq \beta_{i-1}(e_{i-1}(0), t) + \varepsilon_{i-1}'$ and hence concludes the proof. ■

We want to point out again that the main ingredients for this result are the QUAD condition, the sufficient local coupling, and the acyclic structure of the global coupling graph. Exploiting these properties, the recursive procedure presented above provides a *constructive* way to design the local coupling gains for practical cluster synchronization.

V. Numerical Example

The theoretical results presented above are now illustrated on an exemplary setup. In fact, we show that important dynamical systems meet our assumptions. Additionally, we also illustrate here, how we can extend our framework to systems of different dimensions. We consider a setup involving two system classes $M = 2$. Class Σ_1 is governed by $N_1 = 7$ *Chua's circuits*, and class Σ_2 is governed by $N_2 = 5$ *Van der Pol oscillators*. The dynamic equations for Chua's circuit are

$$\begin{bmatrix}\dot{x}_{1i,1}\\ \dot{x}_{1i,2}\\ \dot{x}_{1i,3}\end{bmatrix} = \begin{bmatrix}a_1(-x_{1i,2} - a_5(x_{1i,1}))\\ x_{1i,1} - x_{1i,2} + x_{1i,3}\\ a_4 x_{1i,2}\end{bmatrix},$$

where

$$a_5(x_{1i,1}) = a_2 x_{1i,1} + \frac{1}{2}(a_3 - a_2)(|x_{1i,1}+1| - |x_{1i,1}-1|).$$

The Van der Pol oscillator is described by

$$\begin{bmatrix}\dot{x}_{2i,1}\\ \dot{x}_{2i,2}\end{bmatrix} = \begin{bmatrix}b_1 x_{2i,2} - \frac{b_2}{3}x_{2i,1}^3 - b_3 x_{2i,1}\\ -b_1 x_{2i,1}\end{bmatrix}.$$

The parameters are set to $a_1 = 0.59/0.12$, $a_2 = -0.07$, $a_3 = 1.5$ $a_4 = 0.59/0.162$, $b_1 = 1$, $b_2 = 5$, $b_3 = 5$. In addition, the systems are equipped with input maps I_3, $[I_2\ 0]$ and output maps I_3, $[I_2\ 0]^\top$, respectively, in order to match the dimensions. The systems are now coupled by $W_1 = 1_7^\top \otimes 1_7 - 7I_7$, $W_2 = 1_5^\top \otimes 1_5 - 5I_5$. Note that these local couplings directly satisfy the coupling conditions and in particular (2). Notably, W_1, $W_2 \in \mathbb{W}$, $\lambda_1 = -7$, and $\lambda_2 = -5$. The coupling between different classes is chosen as $W_{21} =$`rand(7,5)`, and $W_{12} = 0$, which hence makes $\mathscr{T}$ an acyclic graph with $T_{12} = 0$ and $T_{21} = 1$. Both, Chua's circuit and the Van der Pol oscillator, have been proven to be QUAD , i.e., the first was shown explicitly, see [17], and the latter implicitly via contraction, see [24]. We have here $c_1 = 0$. Using arbitrary initial values, the above setting is solved in MATLAB using

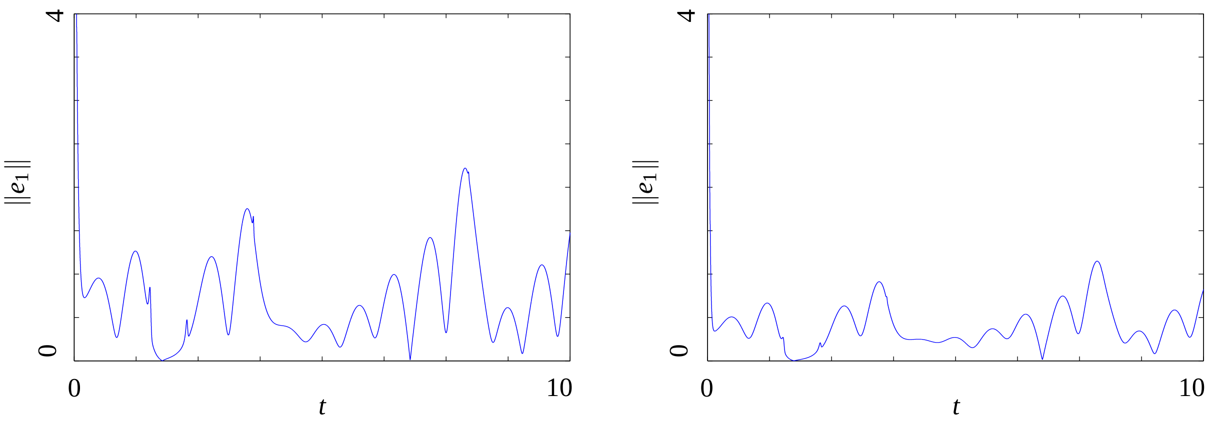

Fig. 2. Error norms for class Σ_1 plotted versus time induced by the respective gains $g_1 = 1$ (left) and $g_1 = 2$ (right). For the higher gain, the errors are just the half of the errors for the low gain, as g_1 influences ε_1 linearly for $c_1 = 0$.

ode45 with two different gains $g_1 = 1$ and $g_1 = 2$. From our results, the error $\|e_1\|$ with the high gain should be less than half the error with the low gain. The plots resulting from our simulation are depicted in Figure 2. One can observe, that the maximal error for the higher gain is just half the error of the lower gain. The simulations clearly support our theoretical findings.

VI. CONCLUSIONS AND FUTURE WORK

We studied practical cluster synchronization for coupled nonlinear systems. Our setup involved several classes of identical systems and we showed, how practical cluster synchronization can be achieved by a suitable choice of local coupling gains. The presented results build upon three main assumptions: (i) the QUAD condition for the dynamical systems, (ii) appropriate couplings between nodes in the same class, and (iii) an acyclic interconnection structure between nodes in different classes. Based on these results, we proposed a constructive recursive procedure to select local coupling gains, such that the synchronization errors of a system class are ultimately bounded by an a-priori chosen bound. The theoretical results have been validated on an exemplary network composed of Van der Pol oscillators and Chua's circuits.

The main challenge for future work is to overcome the restrictive assumption that the global interaction structures follow an acyclic structure.

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