Stabilizing Submanifolds with Passive Input-Output Relations

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***Abstract*— We study submanifold stabilization problems from an input-output perspective. For doing so, we consider feedback interconnections of relations of signals whose squared distance to a given submanifold has finite integral. In our framework, output feedback passivity of the feedforward relation and input strict passivity of the feedback relation with sufficiently large excess of passivity, both with respect to the integral squared distance of their signals to the submanifold under consideration, is sufficient for submanifold stabilization. We show that the distance of the signals in the feedback interconnection to the submanifold remains bounded for bounded exogenous inputs, thus extending the feedback theorem for passive systems to submanifold stabilization problems.**

I. INTRODUCTION

We study control problems in which the output of a plant H_1 must be brought as close to some desired submanifold $M \subset \mathbb{R}^n$ of its output space as the exogenous disturbances ω permit. This is a submanifold stabilization problem and as such includes control problems such as setpoint regulation (in which case M is a singleton), synchronization (in which case M is the span of the vector of ones, cf. [1]), pattern generation (in which case M is a homotopy circle, cf. [2]), and path following (in which case M is the image of a curve, cf. [3]). We study properties of a controller H_2 such that the feedback interconnection of H_1 with H_2, as in Fig. 1, produces signals with bounded integral squared distance to M for bounded exogenous inputs ω.

In this paper, we particularly study the role that passivity plays in such feedback interconnections. We define a meaningful notion of passivity with respect to the integral distance of a signals to M and derive a passivity theorem stating that if H_1 is output feedback passive and $-H_2$ is input strictly passive with a sufficiently large excess of passivity, both with respect to the integral distance of their signals to M, then the feedback interconnection produces signals with bounded integral squared distance to M for bounded exogenous inputs.

We treat H_1 and H_2 as relations on the Cartesian products of their sets of input-output signals. This point of view is similar to Zames [4], [5] and Desoer and Vidyasagar [6]. As integral distances to M are considered herein, our approach also bares similarity to the work on almost invariant subspaces by Basile and Marro [7] and also Willems [8], [9]. In these latter pieces of work, the distance of a signal to a subspace must me kept small by an appropriate choice of the controller. Our goal is similar, but we allow for general embedded submanifolds.

Passivity has recently been employed as a tool for stabilization of closed sets in state space by El-Hawwary and Maggiore [10], extending the work on setpoint stabilization by Byrnes, Isidori, and Willems [11].

In contrast to these state-space methods, passive input-output relations were recently employed by Scardovi et al. [12] to study a submanifold stabilization problem for the particular case where M is the span of the vector of ones.

This paper generalizes the passivity-based input-output approach to submanifold stabilization to general embedded submanifolds and presents a constructive and graphical framework for this class of problems, as our examples illustrate: We apply our findings to the synchronization problem and to the pattern generator problem. In the former case, our results encompass diffusive couplings (cf. [1]). In the latter case, we present a dimension reduction technique which simplifies the control problem.

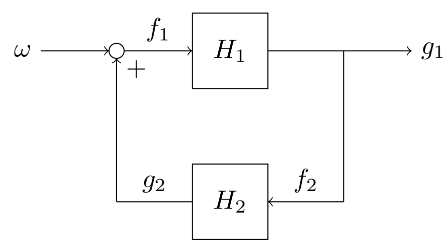

Fig. 1. Feedback interconnection of plant H_1 and controller H_2

II. PROBLEM SETUP

We consider the problem of bringing the output g_1 of a plant H_1 as close to a smoothly embedded submanifold M of its output space $\mathbb{R}^n$ as initial conditions and exogenous disturbances ω admit. In particular, we study conditions on H_2 that let H_1 produce signals that are close to M, whereby H_1 and H_2 are related via the feedback interconnection depicted in Fig. 1, thus satisfying the feedback equations

$$f_1 = g_2 + \omega \tag{1}$$

$$g_1 = f_2 \tag{2}$$

$$(f_1, g_1) \in H_1 \tag{3}$$

$$(f_2, g_2) \in H_2 \tag{4}$$

where f_1 is the input of the plant and g_2 is the output of the controller. We focus on passivity as a sufficient condition on H_1 and H_2 to solve this control problem.

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Classically, H_1 and H_2 are relations on the Cartesian product of

$$\mathscr{L}^2 = \left\{ f : T \to \mathbb{R}^n \,\middle|\, f \text{ measurable}, \int_T \|f(t)\|^2 \, \mathrm{d}\, t < \infty \right\} \tag{5}$$

(or its extension) with itself, where either $T = \mathbb{R}$ or $T = [0, \infty)$. With "measurable" we mean Lebesgue measurable and with $\mathrm{d}\, t$, we mean $\mathrm{d}\, \lambda(t)$, where λ is the Lebesgue measure on T. One then designs H_2 such that the norm

$$\|\cdot\|_{\mathscr{L}^2} : \mathscr{L}^2 \to \mathbb{R}, \;\; f \mapsto \sqrt{\int_T \|f(t)\|^2 \, \mathrm{d}\, t} \tag{6}$$

of g_1 is bounded whenever the norm of ω is bounded. In doing so, it is a convenient fact that $\left(\mathscr{L}^2, \|\cdot\|_{\mathscr{L}^2}\right)$ is a normed vector space which is also complete in the norm $\|\cdot\|_{\mathscr{L}^2}$ (i.e., it is a Banach space). Moreover, with the inner product

$$\langle \cdot, \cdot \rangle : \mathscr{L}^2 \times \mathscr{L}^2 \to \mathbb{R}, \;\; (f, g) \mapsto \int_T f(t) \cdot g(t) \, \mathrm{d}\, t, \tag{7}$$

$\left(\mathscr{L}^2, \langle \cdot, \cdot \rangle\right)$ is a Hilbert space, which is the reason why it receives particular attention (e.g., the passivity property is defined via the inner product).

However, for the submanifold stabilization problem, the distance of g_1 to M – not its norm – must be bounded. Therefore, we consider

$$\mathscr{L}^2_M = \left\{ f : T \to U^\varphi_M \,\middle|\, f \text{ measurable}, \int_T d(f(t), M)^2 \mathrm{d}\, t < \infty \right\} \tag{8}$$

where $d(x, M)$ is defined to be the infimal Euclidean distance of all points in M to x and U^φ_M is a *tubular neighborhood* of M. When φ is a constant, we refer to a tubular neighborhood of the form $U^\varphi_M = \{x \in \mathbb{R}^n | d(x, M) < \varphi\}$.

To introduce tubular neighborhoods, we require some terminology: Given a smoothly embedded submanifold M of $\mathbb{R}^n$, denote its tangent space at $x \in M$ by $\mathcal{T}_x M$. The normal space $\mathcal{N}_x M$ of M at x is defined to be the orthogonal complement of $\mathcal{T}_x M$ in $\mathbb{R}^n$. The normal bundle of M is $\mathcal{N}M = \{(x, y) \in M \times \mathbb{R}^n | y \in \mathcal{N}_x M\}$. Let $U^\varphi_{\mathcal{N}M} = \{(x, y) \in \mathcal{N}M | \, \|y\| < \varphi(x)\}$ and define $\rho : U^\varphi_{\mathcal{N}M} \to \mathbb{R}^n$, $(x, y) \mapsto x + y$ with $U^\varphi_M = \rho\left(U^\varphi_{\mathcal{N}M}\right)$.

Definition 1 (Tubular Neighborhood): The neighborhood U^φ_M of M is said to be a tubular neighborhood of M if it is the diffeomorphic image of $\rho : U^\varphi_{\mathcal{N}M} \to \mathbb{R}^n$.

The reason for restricting ourselves to tubular neighborhoods will become clear later. Note however that $\mathbb{R}^n$ is a tubular neighborhood of $\{0\}$ and hence $\mathscr{L}^2 = \mathscr{L}^2_{\{0\}}$.

Having the notion of $\mathscr{L}^2_M$ at hand, we define

$$\|\cdot\|_{\mathscr{L}^2_M} : \mathscr{L}^2_M \to \mathbb{R}, \;\; f \mapsto \sqrt{\int_T d(f(t), M)^2 \, \mathrm{d}\, t}. \tag{9}$$

Letting H_1 be a relation on $\mathscr{L}^2 \times \mathscr{L}^2_M$ and consequently showing that $\|g_1\|_{\mathscr{L}^2_M}$ is bounded would solve the submanifold stabilization problem.

Example 1: To gain intuition about members of $\mathscr{L}^2_M$, we here present two signals, one of which is in $\mathscr{L}^2_M$ and one of which is not. Let $M = \mathbb{S}$ be the unit sphere. We assume both signals are measurable and of the form

$$f : t \mapsto \begin{bmatrix} (1 + a(t)) \cos(t) \\ (1 + a(t)) \sin(t) \end{bmatrix}. \tag{10}$$

With $|a(t)| < \varphi \leq 1$, f is restricted to the tubular neighborhood $U^\varphi_\mathbb{S} \subset U^1_\mathbb{S}$ of $\mathbb{S}$ ($U^\varphi_\mathbb{S}$ is tubular as $U^1_\mathbb{S}$ is tubular). If now, in addition, a is an $\mathscr{L}^2$ signal, for instance an oscillatory signal with an $\mathscr{L}^2$ amplitude, say $a(t) = \varphi \frac{\sin(t)}{1+t}$, then f is in $\mathscr{L}^2_\mathbb{S}$ as $\mathbb{S}$ has unit radius. This signal is depicted in Fig. 2 (left). If, in contrast, $a \notin \mathscr{L}^2$, for instance an oscillatory signal with an constant amplitude, say $a(t) = (\varphi - \epsilon) \sin(t)$ for some small but positive ϵ, then f would not be in $\mathscr{L}^2_\mathbb{S}$. This signal is depicted in Fig. 2 (right).

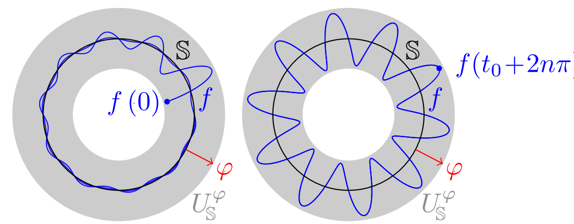

Fig. 2. Examples for signals which are (left) or are not (right) in $\mathscr{L}^2_\mathbb{S}$

The classical input-output approaches employ the extended $\mathscr{L}^2$ space $\bar{\mathscr{L}}^2$, which contains signals that are in $\mathscr{L}^2$ when truncated after any finite time. This allows one to consider "unstable" plants as well. In the same spirit, we do not want to restrict ourselves to plants which produce signals with bounded distance to M and must thus find a meaningful truncation. The *tubular neighborhood theorem* (cf. [13, section II.11] or [14, chapter 10]) will assist us in doing so.

Theorem 1 (Tubular Neighborhood Theorem): If M is a smoothly embedded submanifold, then there exists $\varphi : M \to (0, \infty)$ such that U^φ_M is a tubular neighborhood of M.

In particular, letting $\mathsf{P}_1 : \mathcal{N}M \to M$ be the bundle projection $(x, y) \mapsto x$, it is a consequence of the tubular neighborhood theorem that

$$r = \mathsf{P}_1 \circ \rho^{-1} : \rho\left(U^\varphi_{\mathcal{N}M}\right) \to M \tag{11}$$

is a smooth retraction of the tubular neighborhood onto M (for details on this, we again refer to [13, section II.11] or [14, chapter 10]). For a given $t_0 \in T$, the smooth retraction r allows us to define the truncation

$$f^{t_0}_M(t) = \begin{cases} f(t) & \text{for } t < t_0 \\ r(f(t)) & \text{elsewhere} \end{cases} \tag{12}$$

and with this notion at hand, the extension $\bar{\mathscr{L}}^2_M$ of $\mathscr{L}^2_M$ is given by

$$\bar{\mathscr{L}}^2_M = \left\{ f : T \to U^\varphi_M \,\middle|\, \forall t \in T, \;\; f^t_M \in \mathscr{L}^2_M \right\}. \tag{13}$$

This is consistent with the construction of the extension $\bar{\mathscr{L}}^2$ of $\mathscr{L}^2$.

Example 2: As an example for a function which is contained in $\bar{\mathscr{L}}_M^2$ but not in $\mathscr{L}_M^2$, reconsider the situation from Example 1 with $M = \mathbb{S}$ and $a(t) = (\varphi - \epsilon)\sin(t)$ for some small but positive ϵ. In particular, the truncation of f reads

$$f_{\mathbb{S}}^{t_0} : t \mapsto \begin{cases} \begin{bmatrix} (1 + a(t))\cos(t) \\ (1 + a(t))\sin(t) \end{bmatrix} & \text{for } t < t_0 \\ \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} & \text{elsewhere} \end{cases} \tag{14}$$

where we used that, for $M = \mathbb{S}$, $r : x \mapsto \left({}^{1}/_{\|x\|}\right) x$. As $a^{t_0} \in \mathscr{L}^2$ for any t_0, it follows that $f \in \bar{\mathscr{L}}_{\mathbb{S}}^2$. In Fig. 3, we depict f (left) as well as $f_{\mathbb{S}}^{t_0}$ (right) for some t_0.

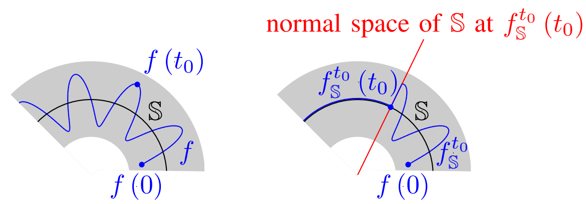

Fig. 3. Example for a signal (left) which is not contained in $\mathscr{L}_{\mathbb{S}}^2$, but whose truncation (right) is, thus letting it be in $\bar{\mathscr{L}}_{\mathbb{S}}^2$

At this point we would like to assure the reader that all presented notions for $\mathscr{L}_M^2$ are consistent with the classical $\mathscr{L}^2$ notions. In particular, if M is $\{0\}$, then we have that $r : x \mapsto 0$ and that hence, the truncation $f_{\{0\}}^t$ is the classical truncation and the extension $\bar{\mathscr{L}}_{\{0\}}^2$ is just the classical extension $\bar{\mathscr{L}}^2$ of $\mathscr{L}^2$. For this reason, we just omit the subindex $\{0\}$ and merely write f^t for $f_{\{0\}}^t$.

It may appear that introducing $\mathscr{L}_M^2$ solves the submanifold stabilization problem, as one is tempted to define all known notions from $\mathscr{L}^2$ for $\mathscr{L}_M^2$ by replacing $\mathscr{L}^2$ with $\mathscr{L}_M^2$ and $\|\cdot\|_{\mathscr{L}^2}$ with $\|\cdot\|_{\mathscr{L}_M^2}$, consequently applying all known results from $\mathscr{L}^2$. However, one encounters issues in doing so. In particular, $\left(\mathscr{L}_M^2, \|\cdot\|_{\mathscr{L}_M^2}\right)$ is not a Banach space, nor a normed vector space, nor even merely a vector space.

If we want to retain these properties, we require a tool representing elements of $\mathscr{L}_M^2$ in $\mathscr{L}^2$. We are again assisted by the tubular neighborhood theorem and the smooth retraction r. The following lemma states that the map

$$\Pi_M : (t \mapsto f(t)) \;\mapsto\; (t \mapsto f(t) - r(f(t))), \tag{15}$$

which also turned out to be useful in [15], indeed allows us to work with elements of $\bar{\mathscr{L}}_M^2$ as if they were in $\bar{\mathscr{L}}^2$.

Lemma 1: For any $f \in \bar{\mathscr{L}}_M^2$, $\Pi_M(f) \in \bar{\mathscr{L}}^2$. In particular, if $f \in \mathscr{L}_M^2$, then $\Pi_M(f) \in \mathscr{L}^2$. Moreover, for any $t \in T$, $\|\Pi_M(f_M^t)\|_{\mathscr{L}^2} = \|f_M^t\|_{\mathscr{L}_M^2}$ and $(\Pi_M(f))^t = \Pi_M(f_M^t)$.

The proof is given in the appendix.

In the light of the lemma, it is correct to indeed refer to the map Π_M as

$$\Pi_M : \bar{\mathscr{L}}_M^2 \to \bar{\mathscr{L}}^2 \tag{16}$$

with the additional property $\Pi_M\left(\mathscr{L}_M^2\right) \subset \mathscr{L}^2$.

Example 3: This example illustrates the map Π_M. Reconsider the situation from Example 1 with $M = \mathbb{S}$ and $a(t) = (\varphi - \epsilon)\sin(t)$ for some small but positive ϵ. As the normal component of f is just a, we have that $\Pi_{\mathbb{S}} f = a$. In Fig. 4, we depict f (left) as well as $\Pi_{\mathbb{S}} f$ (right). We invite keen readers to compare the figure to [8, Fig. 3].

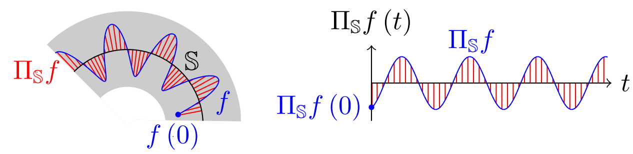

Fig. 4. A signal from $\bar{\mathscr{L}}_{\mathbb{S}}^2$ (left) and its counterpart from $\bar{\mathscr{L}}^2$ (right) resulting from application of $\Pi_{\mathbb{S}}$

Lemma 1 on the map Π_M and the construction of the extension $\bar{\mathscr{L}}_M^2$ reveal why we initially restricted ourselves to tubular neighborhoods; we would have neither been able to define the extension $\bar{\mathscr{L}}_M^2$ nor the map Π_M if we had not imposed this restriction. Further, recalling the tubular neighborhood theorem, this justifies our restriction to submanifolds (in contrast to general subsets of $\mathbb{R}^n$).

Having gathered all these notions, we now recast the feedback equations more precisely. In particular, we treat the plant H_1 as a relation on $\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}_M^2$ and the controller H_2 as a relation on $\bar{\mathscr{L}}_M^2 \times \bar{\mathscr{L}}^2$. This lets the signals on the left-hand side of Fig. 1 live in $\bar{\mathscr{L}}^2$, which is reasonable, as these characterize input energy and disturbances. However, we let the signals on the right-hand side of the feedback interconnection live in $\bar{\mathscr{L}}_M^2$, as we want to bring the output of the plant as close to M as ω admits. This lets the feedback equations read

$$f_1 = g_2 + \omega \in \bar{\mathscr{L}}^2 \tag{17}$$
$$g_1 = f_2 \qquad \in \bar{\mathscr{L}}_M^2 \tag{18}$$
$$(f_1, g_1) \in H_1 \qquad \subset \left(\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}_M^2\right) \tag{19}$$
$$(f_2, g_2) \in H_2 \qquad \subset \left(\bar{\mathscr{L}}_M^2 \times \bar{\mathscr{L}}^2\right). \tag{20}$$

We further require a relation E on $\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}_M^2$ defining the impact that ω has on the output g_1 of the plant H_1, i.e.

$$E = \left\{ (\omega, g_1) \in \left(\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}_M^2\right) \middle| \text{ feedback equations} \right\}. \tag{21}$$

The submanifold stabilization problem is then to find a relation H_2 such that the relation E becomes bounded.

Definition 2: We say that a relation $H \subset \left(\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}_M^2\right)$ is bounded, if for every f in its domain for which there exists $\epsilon \in [0, \infty)$ such that $\|f\|_{\mathscr{L}^2} \leq \epsilon$, there exists $\delta \in [0, \infty)$ such that for all g with $(f, g) \in H$, $\|g\|_{\mathscr{L}_M^2} \leq \delta$.

A bounded relation produces outputs whose distance to M is bounded for inputs whose norm is bounded. Boundedness of E thus precisely characterizes the goal of the submanifold stabilization problem, which we address in the following section.

III. Passivity and Submanifold Stabilization

In the foregoing section, we had constructed the map Π_M in order to work with $\left(\mathscr{L}_M^2, \|\cdot\|_{\mathscr{L}_M^2}\right)$ as if it was the Hilbert space $\left(\mathscr{L}^2, \langle\cdot,\cdot\rangle\right)$. We recall that passive relations on $\bar{\mathscr{L}}^2 \times \bar{\mathscr{L}}^2$ are defined using the inner product $\langle\cdot,\cdot\rangle : \mathscr{L}^2 \times \mathscr{L}^2 \to \mathbb{R}$ (cf. [6, chapter VI]) and thus define passivity in the classical fashion but with the original signals replaced by the Π_M map. As $\Pi_{\{0\}} f = f$, the advantage of this is that we recover the classical passivity definitions for $M = \{0\}$. Consequently, define

$$\mathbb{P}_e^\ell = \Big\{ H \subset \left(\bar{\mathscr{L}}_X^2 \times \bar{\mathscr{L}}_Y^2\right) \Big| \, \forall (f,g) \in H, \, \forall t \in T, \\ \left\langle \Pi_X\left(f_X^t\right), \Pi_Y\left(g_Y^t\right)\right\rangle \geq \ell \left\|g_Y^t\right\|_{\mathscr{L}_Y^2}^2 + e \left\|f_X^t\right\|_{\mathscr{L}_X^2}^2 \Big\} \tag{22}$$

where (X,Y) can be replaced by either $(M,\{0\})$ or $(\{0\},M)$ such that both H_1 and H_2 can be in $\mathbb{P}_e^\ell$. Alike terminology in classical works on passivity, we say that a relation $H \in \mathbb{P}_e^\ell$ is

- passive if $e = 0$, $\ell = 0$,
- output feedback passive (with the lack of passivity ℓ) if $e = 0$, $\ell \in (-\infty, 0)$, and
- input strictly passive (with the excess of passivity e) if $e \in (0,\infty)$, $\ell = 0$.

Input strict passivity is also referred to as coercivity.

Our main result generalizes the classical feedback theorem for passive systems (cf. [6, Section VI.5]) to submanifold stabilization problems: If H_1 is output feedback passive and $-H_2$ is input strictly passive with a sufficiently large excess of passivity, then E is bounded.

Theorem 2: If there exists $\ell \in [0,\infty)$ such that $H_1 \in \mathbb{P}_0^{-\ell}$, then, for any $e \in (\ell,\infty)$, for any $\alpha \in [0,\infty)$, for every $-H_2 \in \mathbb{P}_e^\alpha$, E is bounded.

Proof: It follows from the feedback equations that

$$\left\langle \omega^t, \Pi_M\left(f_{2M}^t\right)\right\rangle = \left\langle f_1^t, \Pi_M\left(g_{1M}^t\right)\right\rangle - \left\langle g_2^t, \Pi_M\left(f_{2M}^t\right)\right\rangle. \tag{23}$$

As $H_1 \in \mathbb{P}_0^{-\ell}$, the inequality

$$\left\langle \omega^t, \Pi_M\left(f_{2M}^t\right)\right\rangle \geq -\ell \left\|g_{1M}^t\right\|_{\mathscr{L}_M^2}^2 - \left\langle g_2^t, \Pi_M\left(f_{2M}^t\right)\right\rangle \tag{24}$$

holds true. With $-H_2 \in \mathbb{P}_e^\alpha$, we further have

$$\left\langle \omega^t, \Pi_M\left(f_{2M}^t\right)\right\rangle \geq -\ell \left\|g_{1M}^t\right\|_{\mathscr{L}_M^2}^2 + \alpha \left\|g_2^t\right\|_{\mathscr{L}^2}^2 + e \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2}^2 \tag{25}$$

and since $g_1 = f_2$, for any $\alpha \in [0,\infty)$, it follows that

$$\left\langle \omega^t, \Pi_M\left(f_{2M}^t\right)\right\rangle \geq (e-\ell) \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2}^2. \tag{26}$$

Next, by the Cauchy-Schwarz inequality,

$$\left\|\omega^t\right\|_{\mathscr{L}^2} \left\|\Pi_M\left(f_{2M}^t\right)\right\|_{\mathscr{L}^2} \geq (e-\ell) \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2}^2. \tag{27}$$

Using the third statement of Lemma 1, we arrive at

$$\left\|\omega^t\right\|_{\mathscr{L}^2} \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2} \geq (e-\ell) \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2}^2. \tag{28}$$

Under the circumstance that $e \in (\ell,\infty)$, this in turn implies

$$\frac{1}{e-\ell} \left\|\omega^t\right\|_{\mathscr{L}^2} \geq \left\|f_{2M}^t\right\|_{\mathscr{L}_M^2}. \tag{29}$$

For any ω in the domain of E for which there exists $\epsilon \in [0,\infty)$ such that $\|\omega\|_{\mathscr{L}^2} \leq \epsilon$, take the limit $t \to \infty$, yielding

$$\frac{\epsilon}{e-\ell} \geq \frac{1}{e-\ell} \|\omega\|_{\mathscr{L}^2} \geq \|f_2\|_{\mathscr{L}_M^2}. \tag{30}$$

Now, for every such ω, choose

$$\delta = \frac{\epsilon}{e-\ell}, \tag{31}$$

revealing that

$$\|f_2\|_{\mathscr{L}^2} \leq \delta \tag{32}$$

and hence proving boundedness of E. ■

We also recover a version of the classical passivity theorem as a special case.

Corollary 1: If $H_1 \in \mathbb{P}_0^0$, then, for any $e \in (0,\infty)$, for every $-H_2 \in \mathbb{P}_e^0$, E is bounded.

Note that all expressions defining passivity here measure distances of signals to M. Treating M as the image of an output of H_1, the presented passivity notion thus resembles incremental passivity [16], where the passivity inequality must hold incrementally for any two input-output tuples, and equilibrium independent passivity [17], wherein the passivity inequality must hold incrementally only with respect to one signal.

Example 4: To illustrate input strict passivity of $-H_2$ graphically, let $M = \mathbb{S}$ and, for simplicity, first consider the condition pointwise in time. Then the condition says that the angle that g_2 encloses with $-\Pi_{\mathbb{S}}(g_1)$ must be acute. The foregoing results thus state that the integrated angle between g_2 and $-\Pi_{\mathbb{S}}(g_1)$ must be acute, i.e. g_2 is allowed to point outside of the tubular neighborhood $U_{\mathbb{S}}^{d(g_1(t),\mathbb{S})}$ of $\mathbb{S}$ as long as it points inside the tubular neighborhood $U_{\mathbb{S}}^{d(g_1(t),\mathbb{S})}$ when integrated. This is depicted in Figure 5.

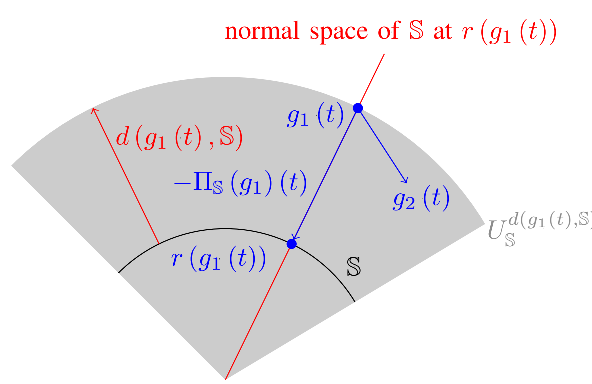

Fig. 5. When integrated, g_2 must make an acute angle with $-\Pi_{\mathbb{S}}(g_1)$ and thus point inside $U_{\mathbb{S}}^{d(g_1(t),\mathbb{S})}$

We wish to remark here that, since all of notions are constructed such that they reduce to classical notions for $M = \{0\}$, we also recover the classical feedback theorem for passive systems and the classical passivity theorem via the substitution $M = \{0\}$. However, we are now also in the position to solve rather general submanifold stabilization problems, as we illustrate on examples in the next section.

IV. Examples

Both synchronization problems and pattern generation can be stated as submanifold stabilization problems, where M is the span of the vector of ones in the former case and a homotopy circle, e.g. the unit circle in the latter case. As we proposed a rather general framework in the foregoing section, we discuss these two particular cases within this section in greater detail.

A. Synchronization

In synchronization or consensus problems, H_1 is a group of systems (i.e. H_1 has a diagonal structure) and their output must be brought to

$$M = \mathcal{S} = \operatorname{span}(1_n) \text{ with } 1_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n \tag{33}$$

(cf. [1]). As input strict passivity of $-H_2$ (with sufficiently large excess of passivity) was shown to be sufficient for boundedness of E in the foregoing section, we infer that input strict passivity of $-H_2$ must be sufficient for synchronization when $M = \mathcal{S}$. To rewrite the input strict passivity condition for $M = \mathcal{S}$, we must first compute $\Pi_{\mathcal{S}}$ for this case. This, in turn, requires computation of r. We recall that $\mathcal{S}$ is a subspace and that hence $\mathbb{R}^n$ is its tubular neighborhood. Further, the retraction r from $\mathbb{R}^n$ onto $\mathcal{S}$ is given by the orthogonal projection onto $\mathcal{S}$, i.e.

$$r : \mathbb{R}^n \to \mathcal{S}, \quad x \mapsto \mathsf{P}_{\mathcal{S}} x = \frac{1}{n} 1_n 1_n^\top x. \tag{34}$$

As an interpretation, r returns the stacked mean of its argument. Consequently, application of $\Pi_{\mathcal{S}}$ to the output g_1 of the plant H_1 yields

$$\Pi_{\mathcal{S}}(g_1) : t \mapsto \left(I - \mathsf{P}_{\mathcal{S}}\right) g_1(t) = \mathsf{P}_{\mathcal{S}}^* g_1(t), \tag{35}$$

where I is the identity matrix, i.e., $\Pi_{\mathcal{S}}(g_1)$ returns the orthogonal projection of g_1 onto the orthogonal complement of $\mathcal{S}$, which is just the stack of deviations of g_1 from its mean, and which is known as the synchronization error in the literature on synchronization. In fact, $\Pi_{\mathcal{S}}(g_1)^\top \Pi_{\mathcal{S}}(g_1)$ is the squared standard deviation of g_1, which is frequently used as a Lyapunov function in synchronization problems. Treating H_2 as a relation on $\mathscr{L}_M^2 \times \mathscr{L}^2$ and thus omitting the technicalities of the extensions $\bar{\mathscr{L}}_M^2 \times \bar{\mathscr{L}}^2$ for simplicity, the characterization of input strict passivity of $-H_2$ then reads that for all $(g_1, g_2) \in H_2$,

$$\int_T -g_2(t) \cdot \mathsf{P}_{\mathcal{S}}^* g_1(t)\, \mathrm{d}\, t \geq e \int_T \left\|\mathsf{P}_{\mathcal{S}}^* g_1(t)\right\|^2 \mathrm{d}\, t. \tag{36}$$

One possible ansatz to solve (36) for g_2 is to assume that H_2 is linear, say

$$g_2 : t \mapsto -K g_1(t). \tag{37}$$

Further, let H_2 satisfy (36) pointwise in time, i.e.

$$K g_1(t) \cdot \mathsf{P}_{\mathcal{S}}^* g_1(t) \geq e \left\|\mathsf{P}_{\mathcal{S}}^* g_1(t)\right\|^2. \tag{38}$$

Under these circumstances, we arrive at the following proposition, wherein the columns of $\mathsf{B}_{\mathcal{S}}^*$ are an orthonormal basis of the orthogonal complement of $\mathcal{S}$, i.e. $\mathsf{B}_{\mathcal{S}}^* \mathsf{B}_{\mathcal{S}}^{*\top} = \mathsf{P}_{\mathcal{S}}^*$, and $K_{\mathtt{sym}}$ denotes the symmetric part of K.

Proposition 1: In the above setting, $-H_2$ is input strictly passive if and only if $\mathsf{B}_{\mathcal{S}}^{*\top} K_{\mathtt{sym}} \mathsf{B}_{\mathcal{S}}^*$ is positive definite, the nullspace of K is contained in $\mathcal{S}$, and $\mathcal{S}$ is an invariant subspace of K.

Proof: We prove necessity and sufficiency separately. First, prove necessity. We start with proving that $\mathsf{B}_{\mathcal{S}}^{*\top} K_{\mathtt{sym}} \mathsf{B}_{\mathcal{S}}^*$ is positive definite. To do so, pick some g_1 such that it attains some value x from the orthogonal complement of $\mathcal{S}$ at time t. Then $\mathsf{P}_{\mathcal{S}} x = 0$ and $\mathsf{P}_{\mathcal{S}}^* x = x$ such that (38) simplifies to

$$\mathsf{P}_{\mathcal{S}}^* x \cdot K_{\mathtt{sym}} \mathsf{P}_{\mathcal{S}}^* x \geq e \left\|\mathsf{P}_{\mathcal{S}}^* x\right\|^2. \tag{39}$$

As the columns of $\mathsf{B}_{\mathcal{S}}^*$ are an orthonormal basis of the orthogonal complement of $\mathcal{S}$, for every x from the orthogonal complement of $\mathcal{S}$, there exists an $y \in \mathbb{R}^{n-1}$ such that $x = \mathsf{B}_{\mathcal{S}}^* y$. Substituting this expression for x and $\mathsf{B}_{\mathcal{S}}^* \mathsf{B}_{\mathcal{S}}^{*\top}$ for $\mathsf{P}_{\mathcal{S}}^*$, we arrive at

$$\mathsf{B}_{\mathcal{S}}^* \mathsf{B}_{\mathcal{S}}^{*\top} \mathsf{B}_{\mathcal{S}}^* y \cdot K_{\mathtt{sym}} \mathsf{B}_{\mathcal{S}}^* \mathsf{B}_{\mathcal{S}}^{*\top} \mathsf{B}_{\mathcal{S}}^* y \geq e \mathsf{B}_{\mathcal{S}}^{*\top} \mathsf{B}_{\mathcal{S}}^* y \cdot \mathsf{B}_{\mathcal{S}}^{*\top} \mathsf{B}_{\mathcal{S}}^* y. \tag{40}$$

Recalling that we picked x arbitrarily from the orthogonal complement of $\mathcal{S}$, the last inequality must hold for all $y \in \mathbb{R}^{n-1}$. As, in addition, $\mathsf{B}_{\mathcal{S}}^{*\top} \mathsf{B}_{\mathcal{S}}^* = I$, the latter implies that $\mathsf{B}_{\mathcal{S}}^{*\top} K_{\mathtt{sym}} \mathsf{B}_{\mathcal{S}}^*$ is positive definite. Next, prove that the nullspace of K is contained in $\mathcal{S}$. To do so, assume for contradiction that there exists an x from the nullspace of K such that $x \notin \mathcal{S}$.

The latter implies that $\left\|x - \mathsf{P}_{\mathcal{S}} x\right\|^2$ is positive, whereas the former implies that $Kx = 0$. This contradicts (38). Last, prove that $\mathcal{S}$ is an invariant subspace of K. For the sake of contradiction, suppose that $K \mathsf{P}_{\mathcal{S}} x \notin \mathcal{S}$ and $\mathsf{P}_{\mathcal{S}}^* x \cdot K \mathsf{P}_{\mathcal{S}} x < 0$. Now decrease $\mathsf{P}_{\mathcal{S}}^* x \cdot K \mathsf{P}_{\mathcal{S}} x$ whilst leaving $\mathsf{P}_{\mathcal{S}}^* x \cdot K \mathsf{P}_{\mathcal{S}}^* x$ constant (this is possible as $\mathsf{P}_{\mathcal{S}} x$ and $\mathsf{P}_{\mathcal{S}}^* x$ can be chosen independently) until (38) is violated, thus revealing that there can exist no such x and that hence $\mathcal{S}$ is an invariant subspace of K. This was the first statement to be proven.

Next, prove sufficiency. To do so, write x as $x = \mathsf{P}_{\mathcal{S}} x + \mathsf{P}_{\mathcal{S}}^* x$. As $\mathsf{P}_{\mathcal{S}} x \in \mathcal{S}$ and $\mathcal{S}$ is an invariant subspace of K, $K \mathsf{P}_{\mathcal{S}} x \in \mathcal{S}$. As $\mathsf{P}_{\mathcal{S}}^* x$ is in the orthogonal complement of $\mathcal{S}$, $\mathsf{P}_{\mathcal{S}}^* x \cdot K \mathsf{P}_{\mathcal{S}} x = 0$, simplifying (38) to (39). As $\mathsf{B}_{\mathcal{S}}^{*\top} K_{\mathtt{sym}} \mathsf{B}_{\mathcal{S}}^*$ is positive definite, there exists an $e \in (0, \infty)$ that satisfies this inequality. This was the last statement to be proven. ■

The proposition reads rather abstract, but we recall that Laplacian matrices of undirected, connected graphs have all the mentioned properties. In particular, the excess of passivity of $-H_2$ is the algebraic connectivity of the graph for this case, which is readily verified by noting that (39) is just the variational characterization of the algebraic connectivity of the graph. This yields the following corollary.

Corollary 2: In the above setting, if K is the Laplacian matrix of an undirected, connected graph, then $-H_2$ is input strictly passive. Moreover, the excess of passivity of $-H_2$ is the algebraic connectivity of the graph.

As Laplacian matrices of connected graphs are regularly used for linear feedback in synchronization problems, and as in many of these problems the algebraic connectivity of the graph must indeed be sufficiently large (cf. [1], [18]), we could recover this known technique in our framework. However, we imposed linearity and pointwise satisfaction as an ansatz. Without this ansatz, one opens the way for recurrently connected graphs, jointly connected graphs, or nonlinear couplings (for the former of the three, keen readers are invited to compare (36) to [19, equation (70)]).

We remark that all results from this section apply to arbitrary subspaces, where $n-1$ must be replaced with the codimension of the subspace.

B. Pattern Generation

In artificial central pattern generators, one must asymptotically stabilize a homotopy circle whilst maintaining an oscillatory behavior on the homotopy circle (cf. [2] and references therein). In our discussion, we will consider the unit sphere

$$M = \mathbb{S} = \{x \in \mathbb{R}^n \,|\, \|x\| = 1\} \tag{41}$$

without loss of generality. Letting H_1 be a passive circuit with oscillatory outputs, in the light of the results from the foregoing section, we wish to find an input strictly passive $-H_2$ in order to solve the pattern generation problem. As the origin can not be retracted to $\mathbb{S}$ (all normal spaces of $\mathbb{S}$ intersect at the origin), the largest tubular neighborhood of $\mathbb{S}$ is $U^1_{\mathbb{S}}$. In this tubular neighborhood, r attains the form

$$r : U^1_{\mathbb{S}} \to \mathbb{S}, \;\; x \mapsto \frac{1}{\|x\|} x. \tag{42}$$

Proceeding and computing $\Pi_{\mathbb{S}}$, one obtains

$$\Pi_{\mathbb{S}}\left(g_1\right) : t \mapsto \left(1 - \frac{1}{\|g_1\left(t\right)\|}\right) g_1\left(t\right). \tag{43}$$

We again treat H_2 as a relation on $\mathscr{L}^2_M \times \mathscr{L}^2$ instead of $\bar{\mathscr{L}}^2_M \times \bar{\mathscr{L}}^2$ to omit technicalities such that the characterization of $-H_2$ being input strictly passive reads that for all $(g_1, g_2) \in H_2$,

$$\begin{aligned} &\int_T -g_2\left(t\right) \cdot \left(1 - \frac{1}{\|g_1\left(t\right)\|}\right) g_1\left(t\right) \mathrm{d}\, t \\ &\quad \geq e \int_T \left\| \left(1 - \frac{1}{\|g_1\left(t\right)\|}\right) g_1\left(t\right) \right\|^2 \mathrm{d}\, t. \end{aligned} \tag{44}$$

We attempt to solve this inequality via a dimension reduction. In particular, we study the function

$$k : T \to \mathbb{R}, \;\; t \mapsto \frac{1}{\|g_1\left(t\right)\| \left(1 - \|g_1\left(t\right)\|\right)} g_1\left(t\right) \cdot g_2\left(t\right), \tag{45}$$

as it "measures" the angle which had to be acute in Example 4 by orthogonally projecting g_2 onto $-\Pi_{\mathbb{S}} g_1$, and note that the condition (44) only depends on the scalar function k and not on all other $n-1$ directions of g_2, as the following proposition states.

Proposition 2: In the above setting, $-H_2$ is input strictly passive if and only if

$$\int_T \left(e - k\left(t\right)\right) d\left(g_1\left(t\right), \mathbb{S}\right)^2 \mathrm{d}\, t \leq 0. \tag{46}$$

Proof: Decompose $g_2\left(t\right)$ into its orthogonal projection onto

$$\operatorname{span}\left(g_1\left(t\right)\right) = \mathcal{N}_{r(g_1(t))}\mathbb{S} \tag{47}$$

given by

$$g_1\left(t\right) \cdot g_2\left(t\right) \frac{1}{\|g_1\left(t\right)\|^2} g_1\left(t\right) = k\left(t\right) \frac{1 - \|g_1\left(t\right)\|}{\|g_1\left(t\right)\|} g_1\left(t\right) \tag{48}$$

and its rejection from $g_1\left(t\right)$, given by

$$\begin{aligned} &g_2\left(t\right) - g_1\left(t\right) \cdot g_2\left(t\right) \frac{1}{\|g_1\left(t\right)\|^2} g_1\left(t\right) \\ =& g_2\left(t\right) - k\left(t\right) \frac{1 - \|g_1\left(t\right)\|}{\|g_1\left(t\right)\|} g_1\left(t\right) \end{aligned} \tag{49}$$

such that, as

$$\Pi_{\mathbb{S}}\left(g_1\right)\left(t\right) = g_1\left(t\right) - \frac{1}{\|g_1\left(t\right)\|} g_1\left(t\right) \in \mathcal{N}_{r(g_1(t))}\mathbb{S}, \tag{50}$$

the rejection vanishes in the inner product on the left-hand side of (44), i.e.

$$\begin{aligned} &-g_2\left(t\right) \cdot \left(g_1\left(t\right) - \frac{1}{\|g_1\left(t\right)\|} g_1\left(t\right) \right) \\ &\quad = k\left(t\right) \left\| g_1\left(t\right) - \frac{1}{\|g_1\left(t\right)\|} g_1\left(t\right) \right\|^2, \end{aligned} \tag{51}$$

simplifying (44) to the equivalent condition (46), which was claimed. ∎

The implication of the foregoing proposition is that the condition (44) on the function g_2 could be broken down to the simpler condition (46) on the scalar function k, which, simply said, states that k must attain sufficiently large values. In the proof of the proposition, we saw that this is due to the fact that normal spaces of $\mathbb{S}$ are one-dimensional, or in other words, that $\mathbb{S}$ has dimension $n-1$ and codimension 1. This, in the same fashion, generalizes to other submanifolds in the sense that the strict passivity condition is always equivalent to a simpler condition on a function having the codimension of the submanifold as its dimension.

V. Conclusion

We studied the submanifold stabilization problem from an input-output perspective. For doing so, we replaced the usual integral squared norm by the integral squared distance to the submanifold. Establishing a passivity framework for relations on the sets containing signals whose truncations have finite integral squared distance to the submanifold, we found that the feedback interconnection of an output feedback passive plant and an input strictly passive controller produces signals of finite integral squared distance to the submanifold for finite exogenous inputs, as long as the excess of passivity of the controller is sufficiently large. In the latter sense, we established a feedback theorem for submanifold stabilization.

APPENDIX: PROOF OF LEMMA 1

Proof: We prove the four statements separately.

First, prove that for any $f \in \bar{\mathscr{L}}^2_M$, $\Pi_M(f) \in \bar{\mathscr{L}}^2$. To see this, recall that $f \in \bar{\mathscr{L}}^2_M$ is defined as $f^{t_0}_M \in \mathscr{L}^2_M$ for any $t_0 \in T$, whereas the latter just means that

$$\int_T d\left(f^{t_0}_M(t), M\right)^2 \mathrm{d}\,t < \infty. \tag{52}$$

As we have that

$$f^{t_0}_M(t) = \begin{cases} f(t) & t < t_0 \\ r(f(t)) & \text{elsewhere,} \end{cases} \tag{53}$$

and as $d(r(f(t)), M) = 0$ within tubular neighborhoods of M, it follows that

$$\int_T d\left(f^{t_0}_M(t), M\right)^2 \mathrm{d}\,t = \int_{(-\infty,t_0)} d(f(t), M)^2 \mathrm{d}\,t, \tag{54}$$

where $(-\infty, t_0)$ must be replaced by $[0, t_0)$ for the case that $T = [0, \infty)$. As $d(f(t), M) = \|f(t) - r(f(t))\|$ within tubular neighborhoods, it further follows that

$$\int_{(-\infty,t_0)} d(f(t), M)^2 \mathrm{d}\,t = \int_{(-\infty,t_0)} \|f(t) - r(f(t))\|^2 \mathrm{d}\,t. \tag{55}$$

From

$$(\Pi_M(f))^{t_0} : t \mapsto \begin{cases} f(t) - r(f(t)) & t < t_0 \\ 0 & \text{elsewhere,} \end{cases} \tag{56}$$

we conclude that

$$\int_{(-\infty,t_0)} \|f(t) - r(f(t))\|^2 \mathrm{d}\,t = \int_T \left\|(\Pi_M(f))^{t_0}(t)\right\|^2 \mathrm{d}\,t, \tag{57}$$

revealing that, for any $t_0 \in T$,

$$\int_T \left\|(\Pi_M(f))^{t_0}(t)\right\|^2 \mathrm{d}\,t < \infty, \tag{58}$$

which is the characterization of $\Pi_M(f)$ being in $\bar{\mathscr{L}}^2$. This was the first statement to be proven.

Next, prove that if $f \in \mathscr{L}^2_M$, then $\Pi_M(f) \in \mathscr{L}^2$. To do so, recall that $f \in \mathscr{L}^2_M$ is defined as

$$\int_T d(f(t), M)^2 \mathrm{d}\,t < \infty. \tag{59}$$

As $d(f(t), M) = \|f(t) - r(f(t))\|$ within tubular neighborhoods,

$$\int_T \|f(t) - r(f(t))\|^2 \mathrm{d}\,t < \infty. \tag{60}$$

As $\Pi_M(f)$ was defined as $t \mapsto f(t) - r(f(t))$, it follows that $\Pi_M(f) \in \mathscr{L}^2$. This was the second statement to be proven.

The identity $\|\Pi_M(f^t_M)\|_{\mathscr{L}^2} = \|f^t_M\|_{\mathscr{L}^2_M}$ is proven just as the first statement, but with $< \infty$ replaced by some constant.

It remains to prove that $(\Pi_M(f))^t = \Pi_M(f^t_M)$, where $\Pi_M\left(f^{t_0}_M\right)$ was defined as

$$\begin{aligned}\Pi_M\left(f^{t_0}_M\right)(t) &= f^{t_0}_M(t) - r\left(f^{t_0}_M(t)\right) \\ &= \begin{cases} r(t) - r(f(t)) & t < t_0 \\ r(f(t)) - r(r(f(t))) & \text{elsewhere.} \end{cases}\end{aligned} \tag{61}$$

Using the identity $r \circ r = r$, we arrive at

$$\Pi_M\left(f^{t_0}_M\right) : t \mapsto \begin{cases} f(t) - r(f(t)) & t < t_0 \\ 0 & \text{elsewhere,} \end{cases} \tag{62}$$

which, comparing with (56), reveals that $(\Pi_M(f))^t$ equals $\Pi_M(f^t_M)$.

This was the last statement to be proven. ■

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