Two Gradient-Based Control Laws on $SE(3)$ Derived from Distance Functions

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Abstract We are interested in gradient systems on the special Euclidean group with application in the control of rigid bodies. This is motivated by the idea of lifting the gradient system to a control law for a systems with Newtonian dynamics, all in the spirit of Daniel Koditschek. In particular, we want to compute gradients of distance functions; in these flows, we can enforce stability of our reference configurations by construction. Therefore, we first outline the computation of a gradient systems on $SE(3)$ on the example of a distance function associated with a Riemannian metric proposed by Frank Park and Roger Brockett. Consequently, we choose a distance function that is easy to compute in camera vision systems and derive the corresponding gradient flow.

1 Introduction

The Lie group $SE(3)$ is of special interest in various applications; amongst them are camera positioning, vehicle trajectory planning, and robot modeling, to name a few. For some of these problems, it is important to generate curves on $SE(3)$ (planning problem). For others, one wants to move a point on $SE(3)$ to another (control problem). Both can essentially be described as the problem of minimizing distances on $SE(3)$ (offline for the former and online for the latter case).

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Here, we want to consider special gradient algorithms based on distance functions to solve planning and control problems on $SE(3)$. More precisely, we consider state feedback control laws for the kinematic equations of a rigid body. The consideration of the kinematic equations is not restrictive in the considered setup, since there are methods in literature to derive a controller for the full equations of motion from the control law for the kinematic equations [5]. We derive the state feedback laws with the help of distance functions. The resulting closed loop vector field is then given by the gradient of the respective distance function, which permits the analysis of the closed loop convergence behavior. We utilize two distance functions. The first distance function is associated with the Riemannian metric proposed by Frank Park and Roger Brockett [6]. the second distance function is one that is particularly easy to compute in camera vision systems.

When choosing coordinate charts, there are established solutions to the above problems [1]. That is, one chooses an appropriate local parametrization, for instance Euler angles or quaternions, to then implement known control procedures in these coordinates. Only, given this, one has to implement a rule that applies when switching coordinate charts. In contrast, working without coordinate charts can be of interest whether because the resulting methods can be more *objective* [2] or just more *natural* [3, 4]. In the past, gradient systems have been used when working without coordinate charts; in particular, gradients of Morse-Bott functions have provided feedback laws with almost global convergence [7, 8, 9, 10].

The remainder of the paper is structured as follows; section 2 entails some preliminaries and the problem formulation, where we explain some basic facts about the special Euclidean group in subsection 2.1 and state the problem in subsection 2.2. In section 3, we present our main results. Therein, subsection 3.1 contains the design procedure for the control law based on the scale-dependent metric and subsection 3.2 contains the design procedure for the control law based on distance function from camera vision. Within both subsections, we split the design procedure into three subsubsections; subsubsections 3.1.1 and 3.2.1 contain the formulations for the distance functions, respectively. In subsubsections 3.1.2 and 3.2.2, we derive the corresponding gradients. Consequently, we propose the associated control laws in subsubsections 3.1.3 and 3.2.3 and investigate their convergence properties. We conclude the paper in section 4.

2 Preliminaries and Problem Statement

In this section, we briefly sketch some facts about the special Euclidean group as well as the problem under investigation.

2.1 Preliminaries

In the following, we collect some background information on the special Euclidean group $SE(3)$. For a detailed exposition of the following we refer to [1, Chapter 2]. The special Euclidean group is the set $SE(3) = \{(R,d) | R \in SO(3), d \in \mathbb{R}^3\}$ together with the group operation $(R_1,d_1)(R_2,d_2) \mapsto (R_1R_2, R_1d_2 + d_1)$, where $SO(3) = \{R \in \mathbb{R}^{3\times3} | R^{-1} = R^\top, \det R = 1\}$ is the set of rotation matrices. The tangent space of $SO(3)$ at a point $R \in SO(3)$ is given by $\mathrm{T}_R SO(3) = \{\xi \in \mathbb{R}^{3\times3} | \xi = R\Omega, R \in SO(3), \Omega = -\Omega^\top\}$. The Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is given by $\mathfrak{so}(3) = \mathrm{T}_I SO(3)$, which are the skew-symmetric matrices. For a $\mathbb{R}^3$ association of a $\mathfrak{so}(3)$ element, we can use the natural function $Q : \mathfrak{so}(3) \to \mathbb{R}^3$ given through $Q(\Omega) \times x = \Omega x$, where $\times$ is the cross-product. As a consequence, the Lie algebra $\mathfrak{se}(3)$ of $SE(3)$ is given by (Ω, v) where $\Omega \in \mathfrak{so}(3)$ and $v \in \mathbb{R}^3$.

A compact and common notation for the elements of $SE(3)$ which we also utilize here is the so-called homogeneous representation, where we write tuples (R,d) as matrices H given through

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in SE(3). \tag{1}$$

The group operation then corresponds to matrix multiplication. In a similar fashion, we can also represent the element $(\Omega, v) \in \mathfrak{se}(3)$ as matrices, i.e.

$$T = \begin{bmatrix} \Omega & v \\ 0 & 0 \end{bmatrix} \in \mathfrak{se}(3). \tag{2}$$

By this, the tangent space of $SE(3)$ at a point $H \in SE(3)$ is given by $\mathrm{T}_H SE(3) = \{V \in \mathbb{R}^{4\times4} | V = HT, H \in SE(3), T \in \mathfrak{se}(3)\}$, when using matrix notation, and we will refer to its elements as

$$V = \begin{bmatrix} \xi & \zeta \\ 0 & 0 \end{bmatrix} \in \mathrm{T}_H SE(3). \tag{3}$$

Thus, $SE(3)$ is invariant with respect to every dynamical system of form $\dot{H} = HT$ with $H \in SE(3)$ and $T \in \mathfrak{se}(3)$. One refers to such equations as the kinematic equations and T underlies a dynamical system itself, which is referred to as the dynamic equations. Most generally, the control input is applied to these dynamic equations. However, one can as well assume to have T as the control input as one can derive a suitable input for the dynamic equations for every choice of T [5].

2.2 Problem statement

We consider control systems on $SE(3)$ of the form

$$\dot{H} = HU, \tag{4}$$

where $H \in SE(3)$ is the state of the system and $U \in \mathfrak{se}(3)$ is an element of the Lie algebra, making $\dot{H}$ an element of $\mathrm{T}_H SE(3)$. The problem we want to solve is the following; find a state-feedback law of the form

$$U = U(H, H^*), \tag{5}$$

such that the closed loop converges to the reference $H^* \in SE(3)$ for almost any initial condition $H \in SE(3)$ and such that H^* is stable. Although mechanical control systems usually entail Newtonian dynamics, the above control problem is of interest for such systems [5]. This is because one can derive a control law for the system with Newtonian dynamics from the control law for the system with dynamics (4).

3 Main results

In the following we want to derive state feedback laws to solve the problem described in Section 2.2. We utilize a three-step procedure to derive the feedback law. In the first step, we define a distance function which measures the distance between our initial configuration H and the desired configuration H^*. In the second step, we compute the differential of this distance function which we utilize in the third step to derive a feedback (5) such that the closed loop vector field is the gradient of the distance function. In the subsequent discussion, we show that the resulting closed loop has the desired convergence properties. We carry out these computations for two different distance functions which result in two different closed loop systems. The first one is discussed in Section 3.1 and the second one in Section 3.2.

3.1 A Gradient-Based Controller Based on the Scale-Dependent Metric

This subsection is dedicated to a control law derived from the gradient of a particular distance function based on the left-invariant Riemannian metric proposed by Frank Park and Roger Brockett, commonly referred to as the scale-dependent metric. We subsequently derive the distance function, the corresponding gradient, and the associated control law.

3.1.1 The Distance Function for the Scale-Dependent Metric

In this section, we compute the distance function based on the Riemannian metric proposed in [6]. The metric structure both of $SO(3)$ and $SE(3)$ is of interesting nature and has been investigated excessively, especially by Frank Park [6, 11], among others. In particular, left-invariant Riemannian metrics are of interest in application, as they are independent of inertial coordinates [2], which are the coordinates placing $\mathbb{R}^3$ in $\mathbb{E}^3$. Herein, we will thus rely on the left-invariant Riemannian metric $\langle\cdot,\cdot\rangle : \mathrm{T}_H SE(3)\times \mathrm{T}_H SE(3)\to\mathbb{R}$ proposed by Park and Brockett [6] given through

$$\langle V,V^*\rangle = \begin{bmatrix} Q\left(R^\top\xi\right) \\ R^\top\zeta \end{bmatrix}^\top \begin{bmatrix} \alpha I & 0 \\ 0 & \beta I \end{bmatrix} \begin{bmatrix} Q\left(R^\top\xi^*\right) \\ R^\top\zeta^* \end{bmatrix}, \tag{6}$$

where $\begin{bmatrix} Q\left(R^\top\xi\right) & R^\top\zeta \\ 0 & 0 \end{bmatrix} = T$ and $\begin{bmatrix} Q\left(R^\top\xi^*\right) & R^\top\zeta^* \\ 0 & 0 \end{bmatrix} = T^*$; The procedure of bringing tangent elements to the identity by group multiplication and then calculating the Riemannian metric with elements of the Lie algebra is common on Lie groups. In mechanics, the resulting notion is called the *twist* $T = H^{-1}V$ ($H\in SE(3)$, $V\in \mathrm{T}_H SE(3)$, and $T\in\mathfrak{se}(3)$). The Riemannian metric (6) is called the *scale dependent metric*. This if of interest as the scale-dependence vanishes as we progress with our design procedure, an effect that can be interesting in practice [12, 13, 14].

The geodesics $\Gamma:\mathbb{R}\to SE(3)$, $s\mapsto\Gamma(s)$ between H and H^* associated with (6) are found by minimizing the functional $\int_0^1\langle\frac{\mathrm{d}}{\mathrm{d}s}\Gamma,\frac{\mathrm{d}}{\mathrm{d}s}\Gamma\rangle\,\mathrm{d}s$ over all curves joining H and H^* such that $\Gamma(0)=H$ and $\Gamma(1)=H^*$. Calculating geodesics can in general turn out to be tedious and is not within the scope of this paper. We therefore omit the precise calculation and instead refer to Park [11] for details. To sketch the calculation, we only want to mention that geodesics on $SE(3)$ between H and H^* associated with the Riemannian metric (6) can be obtained from the geodesics in $SO(3)$ and $\mathbb{R}^3$, yielding

$$\Gamma(s) = \begin{bmatrix} R\exp\left(\log\left(R^\top R^*\right)s\right) & d+s\left(d^*-d\right) \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where $\log: SO(3)\to\mathfrak{so}(3)$ and $\exp:\mathfrak{so}(3)\to SO(3)$ are the logarithmic and the exponential map, respectively. Notably, (7) is the one-parameter family of screw motions.

The twist T of Γ is given through the formula

$$T(s) = \Gamma(s)^{-1}\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s), \tag{8}$$

and we have

$$T(s) = \left(\log\left(R^\top R^*\right), \left(\exp\left(\log\left(R^\top R^*\right)s\right)\right)^\top R^\top\left(d^*-d\right)\right). \tag{9}$$

Applying the Riemannian metric (6) to the twist (9), we have

$$\langle\frac{\mathrm{d}\Gamma(s)}{\mathrm{d}s},\frac{\mathrm{d}\Gamma(s)}{\mathrm{d}s}\rangle = \alpha\left(Q\left(\log\left(R^\top R^*\right)\right)\right)^\top Q\left(\log\left(R^\top R^*\right)\right) + \beta\left(d^*-d\right)^\top\left(d^*-d\right),$$

which we integrate over the interval $[0,1]$. Then, applying the useful identity $2Q(\Omega)^\top Q(\Omega) = \mathrm{tr}\left(\Omega^\top\Omega\right)$ for $\Omega\in\mathfrak{so}(3)$, we arrive at our distance function $d: SE(3)\times SE(3)\to\mathbb{R}$ given by

$$d^2(H,H^*) = \frac{\alpha}{2}\operatorname{tr}\left(\left(\log\left(R^\top R^*\right)\right)^\top \log\left(R^\top R^*\right)\right) + \beta\left(d^* - d\right)^\top\left(d^* - d\right), \quad (10)$$

which just agrees fine with the distance function obtained in [11].

3.1.2 The Gradient for the Scale-Dependent Metric

We are hence in the position to describe our error function $e : SE(3) \to \mathbb{R}$ as the distance between H and H^*, meaning e is the same function as d, only for fixed H^*. This is just writing $e(H) = d^2(H,H^*)$. We can therefore formulate our design goal as the the optimization problem

$$\begin{aligned} &\text{minimize} \;\; e(H) \\ &\text{subject to} \;\; H \in SE(3). \end{aligned} \quad (11)$$

Next, as we have taken e as our "cost", we have to compute the direction in which e decreases, i.e. the tangent element of $SE(3)$ which is gradient of e. For doing so, we first need the directional derivative of e at $V \in \mathrm{T}SE(3)$, which is

$$\mathrm{d}_H e(V) = \left.\frac{\mathrm{d}}{\mathrm{d}\gamma} e \circ A(\gamma)\right|_{\gamma=0}, \quad (12)$$

where $A : [-\varepsilon,\varepsilon] \to SE(3)$ is such that $A(0) = H$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma}A(\gamma) = V$. Writing $A(\gamma) = (R_A(\gamma), d_A(\gamma))$ and $V = (\xi,\zeta)$, we have

$$\begin{aligned} &\alpha\operatorname{tr}\left(\left(\log\left(R_A^\top(\gamma)R^*\right)\right)^\top R^{*\top}R_A(\gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}R_A^\top(\gamma)R^*\right) + 2\beta\left(d^* - d_A(\gamma)\right)^\top\left.\left(-\frac{\mathrm{d}}{\mathrm{d}\gamma}d_A(\gamma)\right)\right|_{\gamma=0} = \\ =&\alpha\operatorname{tr}\left(\left(\log\left(R^\top R^*\right)\right)^\top R^{*\top}R\xi^\top R^*\right) + 2\beta\left(d^* - d\right)^\top(-\zeta) = \mathrm{d}_H e(V). \end{aligned} \quad (13)$$

To obtain the gradient, one has to apply the formula

$$\langle \operatorname{grad} e(H), V\rangle = \mathrm{d}_H e(V), \quad (14)$$

where V is assumed to be tangent to the same element, that $\operatorname{grad} e(H)$ is tangent to, and to solve for $\operatorname{grad} e(H)$. Using the representation $\operatorname{grad} e(H) = \left(\xi_{\text{grad}}, \zeta_{\text{grad}}\right)$, we arrive at

$$\frac{\alpha}{2}\operatorname{tr}\left(\xi_{\text{grad}}^\top \xi\right) + \beta\zeta_{\text{grad}}^\top\zeta = \mathrm{d}_H e(V). \quad (15)$$

Knowing that the trace is invariant under both, cyclic permutations and transposing, and applying the rule $(\log(R))^\top = -\log(R)$ for $R \in SO(3)$, we are able to equate coefficients between (13) and (15) to get

$$\operatorname{grad} e(H) = \begin{bmatrix} -2R^*\log\left(R^\top R^*\right)R^{*\top}R & -2\left(d^* - d\right) \\ 0 & 0 \end{bmatrix}. \quad (16)$$

3.1.3 The Control Law for the Scale-Dependent Metric

We arrive at the dynamical system $\dot{H} = -\operatorname{grad} e(H)$. That is

$$\dot{H} = \begin{bmatrix} 2R^* \log\left(R^\top R^*\right) R^{*\top} R & 2\left(d^* - d\right) \\ 0 & 0 \end{bmatrix}, \tag{17}$$

similarly to the results of Bullo and Murray [15], and we will consequently investigate the stability of (17).

Remark 1. If we want to include time-dependence of H^* explicitly (in the sense that it is a reference signal), and if $H^*(t)$ is sufficiently smooth, we could repeat the last steps of our derivation to get

$$\dot{H} = \begin{bmatrix} 2R^* \log\left(R^\top R^*\right) R^{*\top} R - R^* \dot{R}^{*\top} R & 2\left(d^* - d\right) + \dot{d}^* \\ 0 & 0 \end{bmatrix} \tag{18}$$

instead of (17).

Theorem 1. *The equilibrium $H = H^*$ of* (17) *is asymptotically stable.*

Proof. Suppose the Lyapunov function candidate $W(H) = e(H)$. We have W positive semidefinite because d is a distance function. Further, W is zero iff $H = H^*$. We take the directional derivative

$$\dot{W}(H) = \frac{\alpha}{2} \operatorname{tr}\left(\left(R^{*\top} R \dot{R}^\top R^*\right)^\top \log\left(R^\top R^*\right) + \left(\log\left(R^\top R^*\right)\right)^\top R^{*\top} R \dot{R}^\top R^* \right) - 2\beta \dot{d}^\top \left(d^* - d\right),$$

and, substituting (17) into (3.1.3), we have

$$\dot{W}(H) = 2\alpha \operatorname{tr}\left(\left(\log\left(R^\top R^*\right)\right)^2 \right) - 4\beta \left(d^* - d\right)^\top \left(d^* - d\right), \tag{19}$$

which equals $\dot{W}(H) = -4W(H)$ and means that $\dot{W}(H)$ is negative semidefinite and zero if $H = H^*$. □

3.2 A Gradient-Based Controller Based on a Distance Function from Camera Vision

Again, we split the subsection into three parts. First, we define our distance function. Then, we take the gradient with respect to one of its arguments. Consequently, we define our control law accordingly and investigate its convergence properties.

3.2.1 The Distance Function from Camera Vision

We could see that the gradient flow of the distance function (10) computed above had some nice convergence properties. However, to compute the distance function

(10) one has to compute the rotation matrices of the current and the desired configuration, respectively. Instead, in camera vision systems, it is common to only know the *position* of some characteristic points of the rigid body. These points are usually captured with camera markers, e.g. retroflective or colored markers. In such settings, the distance between the current and the desired position of the markers

$$d^2(H,H^*) = \sum_{b\in M} b^\top (R-R^*)^\top (R-R^*)\, b + (d-d^*)^\top (d-d^*) \tag{20}$$

where M is the set of marker positions in body-fixed coordinates, appears to be an appropriate distance function. In particular, it approximates the volume enclosed by the body particles between current and desired position

$$d^2(H,H^*) = \int_B b^\top (R-R^*)^\top (R-R^*)\, b \,\mathrm{d}b + (d-d^*)^\top (d-d^*), \tag{21}$$

where B is the set of body particles in body-fixed coordinates. We now mimic the steps taken to arrive at (17).

3.2.2 The Gradient for the Distance Function from Camera Vision

First, we define $e(H) = d^2(H,H^*)$ and formulate the optimization problem (11) to then compute the directional derivative

$$\mathrm{d}_H\, e(V) = -\int_B b^\top \left(R^{*\top}\xi + \xi^\top R^*\right) b \,\mathrm{d}b + 2(d-d^*)^\top \zeta \tag{22}$$

and apply $\mathrm{d}_H\, e(V) = \langle \operatorname{grad} e(H), V\rangle$. We thus have

$$-\int_B b^\top \left(R^{*\top}\xi + \xi^\top R^*\right) b \,\mathrm{d}b + 2(d-d^*)^\top \zeta = \alpha Q\left(R^\top \xi_{\mathrm{grad}}\right)^\top Q\left(R^\top \xi\right) + \beta \left(R^\top \zeta_{\mathrm{grad}}\right)^\top \left(R^\top \zeta\right). \tag{23}$$

Equating coefficients for ζ, we have $\zeta_{\mathrm{grad}} = \frac{2}{\beta}(d-d^*)$. Equating what remains, we arrive at

$$-\int_B b^\top \left(R^{*\top}\xi + \xi^\top R^*\right) b \,\mathrm{d}b = \frac{\alpha}{4} \operatorname{tr}\left(\xi_{\mathrm{grad}}^\top \xi + \xi^\top \xi_{\mathrm{grad}}\right), \tag{24}$$

where we have used the identities $2Q(\Omega_1)^\top Q(\Omega_2) = \operatorname{tr}\left(\Omega_1^\top \Omega_2\right)$, $\Omega_1, \Omega_2 \in \mathfrak{so}(3)$ and $2\operatorname{tr}(A) = \operatorname{tr}\left(A + A^\top\right)$. We now suppose that ξ and ξ_{grad} are both tangent to R and hence use the ansatz $\xi = R\Omega$, $\xi_{\mathrm{grad}} = R\Omega_{\mathrm{grad}}$. This yields

$$\int_B b^\top \left(\Omega R^\top R^* - R^{*\top} R \Omega\right) b \,\mathrm{d}b = -\frac{\alpha}{4} \operatorname{tr}\left(\Omega_{\mathrm{grad}}\Omega + \Omega\Omega_{\mathrm{grad}}\right). \tag{25}$$

It follows by some tedious computations that

$$\Omega_{\mathrm{grad}} = \frac{2}{\alpha} \int_B \Omega_1 b^\top \Omega_1 R^\top R^* b + \Omega_2 b^\top \Omega_2 R^\top R^* b + \Omega_3 b^\top \Omega_3 R^\top R^* b \,\mathrm{d}b \tag{26}$$

satisfies (25) for all $\Omega \in \mathfrak{so}(3)$, where Ω_1, Ω_2, Ω_3 are the generators of the algebra $\mathfrak{so}(3)$ given by $\Omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $\Omega_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$, and $\Omega_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Consistency between (25) and (26) can however be checked by substituting (26) back into (25). This turns out to be true for all $\Omega \in \mathfrak{so}(3)$.

3.2.3 The Control Law for the Distance Function from Camera Vision

Interestingly, the gradient flow of (21) is, in contrast to the gradient flow of (10), *not* scale-independent. Instead of (17), we hence have

$$\dot{H} = \begin{bmatrix} -\frac{2}{\alpha} R \int_B \sum_{i=1}^{3} \Omega_i b^\top \Omega_i R^\top R^* b \,\mathrm{d}b & -\frac{2}{\beta}(d - d^*) \\ 0 & 0 \end{bmatrix} \tag{27}$$

and we are consequently interested in the stability of the equilibrium $H = H^*$ of (27).

Theorem 2. *The equilibrium $H = H^*$ of* (27) *is asymptotically stable.*

Proof. Consider the Lyapunov function candidate $W(H) = e(H)$. W is positive semidefinite because e is a distance function and $W(H) = 0$ iff $H = H^*$. Now, taking the directional derivative, we have

$$\dot{W}(H) = \int_B -b^\top \left(R^{*\top} \dot{R} + \dot{R}^\top R^* \right) b \,\mathrm{d}b + 2(d - d^*)\dot{d}. \tag{28}$$

Substituting (27), this is

$$\dot{W}(H) = \frac{2}{\alpha} \int_B b^\top \Big(R^{*\top} R (\sum_{i=1}^{3} \Omega_i b^\top \Omega_i R^\top R^* b) + (\sum_{i=1}^{3} \Omega_i b^\top \Omega_i R^\top R^* b)^\top R^\top R^* \Big) b \,\mathrm{d}b - \frac{4}{\beta}(d - d^*)^\top (d - d^*)$$

and we substitute $b^\top R^{*\top} R \Omega_i b = \xi_i$ to see that the above is in fact

$$\dot{W}(H) = -\frac{2}{\alpha} \int_B 2\xi_1^2 + 2\xi_2^2 + 2\xi_3^2 \,\mathrm{d}b - \frac{4}{\beta}(d - d^*)(d - d^*) \tag{29}$$

which satisfies $\dot{W}(H) \leq 0$ and $\dot{W}(H) = 0$ if $H = H^*$. $\square$

4 Conclusion

Inspired by the special properties of distance functions we have computed two gradient systems on the special Euclidean group with the intention to use them as control laws for rigid bodies under Newtonian dynamics. In the first case, we have chosen a distance function that we derived from the scale-dependent metric of Frank Park and Roger Brockett. We found that the resulting system has an asymptotically stable equilibrium at the reference configuration. Subsequently, we mimicked this approach with a distance function that is particularly suited for computation in camera vision systems. Again, we could find that the corresponding gradient system had nice convergence properties; the reference configuration is an asymptotically stable equilibrium.

Open topics include the reduction of the number of feedback variables, inclusion of joint and workspace constraints, as well as the formulation of our control laws for systems under Newtonian dynamics.

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