Periodic Event-Triggered Control for Networked Control Systems Based on Non-Monotonic Lyapunov Functions ⋆

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Abstract

This article considers exponential stabilization of linear Networked Control Systems with periodic event-triggered control for a given network specification in terms of a maximum number of successive dropouts and a constant transmission delay. Based on stability results using non-monotonic Lyapunov functions for discontinuous dynamical systems, two sufficient results for stability of the general model of a linear event-triggered Networked Control System are derived. Those results are used to derive robust periodic event-triggered control strategies. First, a static triggering mechanism for the case without delay is derived. Afterwards, two dynamic triggering mechanisms are developed for the case without and with delay. It is shown how a degree of freedom, being contained in the dynamic triggering mechanisms, can be used to shape the resulting network traffic. The applied adaption technique is motivated by existing congestion control mechanisms in communication networks. The properties of the derived mechanisms are illustrated in a numerical example.

Key words: Networked control systems, event-triggered control, fault-tolerant.

1 Introduction

Control systems, in which some or all links in the feedback loop are replaced by a shared communication medium, are known as Networked Control Systems (NCS). Analysis of NCS has been an active field of research in recent years. An overview about early results can be found in [14]. Due to the limited bandwidth in a shared communication medium, it is natural to search for sampling strategies that are able to reduce the transmission frequency for a given control task. Event-based sampling schemes address this objective by sampling and sending messages only when necessary for guaranteeing desired properties of the control system. Typically the main challenge for designing an event-based sampling scheme is the derivation of an appropriate, often state-dependent, trigger rule that indicates the need for a transmission. An introduction to and an overview over standard approaches can be found in [13].

While the motivation for using event-based sampling strategies for NCS is intuitive, the actual combination of imperfect communication and aperiodic sampling is less trivial. The first investigations about the combination were conducted in a stochastic setup numerically in [4] and analytically for an integrator system in [23]. Another analytical study in [3], highlights that the choice of the communication protocol needs to be made in conjunction with the choice of the sampling strategy. Discrete-time nonlinear systems were investigated in [22], where conditions for stochastic stability, depending on the packet loss rate, a bound on the open loop growth rate, and the computational capabilities of the actuator are derived. The recent work in [5] has jointly considered random dropouts and malicious attacks. The derived condition for almost sure asymptotic stability relates an asymptotic dropout rate with a closed-loop decrease and open loop increase of a Lyapunov-like function, which is conceptually related to the type of Lyapunov functions used in the work at hand for deterministic stability results.

In order to establish guaranteed stability in the sense of Lyapunov for NCS, alternative dropout models have

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been considered in the NCS literature. In those models, the dropout process is typically described by the maximal number of successive dropouts. While stabilization with such a model was considered in [29] for time-triggered communication, the first study for event-triggered communication with a bounded number of successive dropouts was given in [27]. Therein, a relation between the threshold in a static trigger rule and the number of successive dropouts is computed, such that a less conservative static trigger rule, that is sufficient for guaranteeing stability, is never violated although packet dropouts occur. An alternative approach was presented in [11]. It exploits the boundedness of packet dropouts such that a retransmission is successful at latest after a known number of attempts. Thus, this approach forces to retransmit after a packet loss without reevaluating the trigger condition and treats packet dropouts as a prolongation of the delay. In [19], robustness of a partially event-based and probabilistic scheduling mechanism to a similar dropout model, given by a finite number of losses in every finite window, was investigated for a multi-loop NCS. Recently, a general approach for dealing with bounded packet dropouts and nonlinear systems in continuous event-triggered control with time regularization, i.e., a setup where the triggering condition is monitored continuously as soon as a certain designed time after the last transmission has elapsed, has been described in [7]. Therein, different output based triggering strategies are derived that are able to deal with network protocols with and without acknowledgement mechanisms. Furthermore, the article presents static and dynamic triggering mechanisms. While in static trigger rules a certain error, induced by the event-based sampling mechanism, is monitored and compared to the state or output of the system with a fixed threshold, the threshold can be dynamically generated in dynamic event-triggered control, see [10]. The results in [7] show that dynamic event-triggered strategies are able to significantly reduce the resulting network load.

One important progress for applying event-based sampling schemes to NCS was made by the concept of periodic event-triggered control (PETC), introduced for linear systems without network induced imperfections in [12]. PETC describes the idea to derive a trigger condition that is monitored with a given baseline sampling period while one can give guarantees for the behavior of the original continuous-time system. This approach has the advantage that all resulting inter-event times are multiples of the baseline sampling period, which can be important for scheduling many applications on a shared communication medium. An investigation of NCS with linear plants subject to bounded packet dropouts and in particular to bounded but time-varying transmission delays was presented in [21]. A static trigger rule is derived based on a suitably chosen Lyapunov-Krasovskii functional for the unknown time-varying delay. In a next step, packet dropouts are essentially handled via the same idea as in [27], i.e., by lowering the static threshold such that the original trigger rule is never violated although a bounded number of successive packets can be dropped.

In a preliminary study for the work at hand [17], an alternative approach for packet dropouts in a PETC setup was investigated. The main idea is to omit the hard requirement that the Lyapunov function needs to decrease at every sampling instant to allow for a slight increase during times when packet loss occurs. Therefore, an alternative discretization of the system, based on the time instants when information is successfully transmitted, is employed to derive a static trigger rule. This static trigger rule is designed to guarantee exponential stability for the alternatively discretized system and the study showed that one can conclude exponential stability for the original continuous-time system from that. This alternative discretization leads to a similar behavior of a Lyapunov function candidate for the original system as the described idea in the stochastic setup of [5] and the approach for continuous event-triggered control without dropouts in [28]. In [17], it is already shown that this behavior can be analyzed with the help of stability theory for discontinuous dynamical systems based on non-monotonic Lyapunov functions as presented in [20]. The idea of non-monotonic Lyapunov functions has shown to be helpful in other areas as well, e.g. in [1] for simplifying the computational search for Lyapunov functions as well as in [9] and [15] for optimization based stabilization of NCS.

The work at hand extends, improves, and generalizes our preliminary study of [17]. It uses the stability result from [20] to derive two general results that simplify analysis and design of event-triggered control systems with bounded packet loss. Those results characterize the main conditions for stabilization as uniform lower and upper bounds on the inter-event intervals and the decrease of a Lyapunov function between two successful transmissions. They are in general applicable to a setup with constant delays, while in [17] no transmission delay is considered. The study at hand derives static and dynamic triggering strategies based on those general results. The dynamic strategy is furthermore extended to the case with constant delays and an adaption mechanism is derived that shapes the resulting network traffic to be better applicable for a shared communication medium. Thus, the main contributions compared to the preliminary results in [17] and the available literature are; (1) the general derivation of two results based on non-monotonic Lyapunov functions that simplify analysis and design of event-based sampling mechanisms for NCS with bounded packet dropouts; (2) the derivation of a static trigger rule based on one of this general results which is modified compared to [17] by avoiding the use of an additional trigger rule; (3) the derivation of dynamic triggering mechanisms for the case with and without delay which are to the best of our knowledge the first dynamic event-triggering mechanisms for PETC with

packet dropouts; (4) and an adaptive algorithm that adjusts the dynamic triggering mechanisms such that the resulting network traffic is better suited for communication over a shared medium.

The remainder is organized as follows. The problem setup is described in Section 2 and the general results for analyzing stability are derived in Section 3. A static triggering mechanism is derived in Section 4, while Section 5 considers dynamic triggering mechanisms. A simulation example is presented in Section 6 and Section 7 concludes the article.

1.1 Notation

In the following, $\mathbb{R}_{\geq a}$ ($\mathbb{R}_{>a}$) denotes the set of real numbers that are greater or equal (greater) than, a. Furthermore, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The notation t^- is used as $t^- := \lim_{s<t, s\to t} s$. For a vector x, $\|x\|$ denotes its Euclidean norm while for a matrix A, $\|A\|$ denotes its spectral norm. For symmetric matrices P, Q, the notation $P \succeq Q$ ($P \succ Q$) is used to write that $P - Q$ is positive semidefinite (positive definite), while 0 and I denote zero and identity matrices of suitable dimensions.

2 Problem setup

In this section, we present the setup of the paper and formalize the control objective. The setup is presented and modeled in a quite general fashion to show the class of systems for which the conditions that will be derived in Section 3 can be applied. In the second subsection, we show how the precise network and triggering setup of this article can be modeled.

2.1 General setup

We consider a linear, time-invariant, stabilizable system

$$\dot{x} = Ax + Bu \tag{1}$$

where the state $x : \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ is the solution of (1) for $t \geq t_0$ with $x(t_0) = x_0$ and the input $u : \mathbb{R}_{\geq t_0} \to \mathbb{R}^p$ generated by

$$u = K\hat{x}, \tag{2}$$

$K \in \mathbb{R}^{p\times n}$ and $\hat{x} : \mathbb{R}_{\geq t_0} \to \mathbb{R}^n$ being an estimation of x generated at the actuator with some initial condition $\hat{x}(t_0) = \hat{x}_0 \in \mathbb{R}^n$. The time instants, when current or delayed state information is received at the actuator, are given by the infinite sequence $(\tau_k)_{k\in\mathbb{N}}$ and define a discrete set

$$\mathcal{T} := \{\tau_1, \tau_2, \dots\}. \tag{3}$$

The sequence depends on a state-dependent trigger rule, the design of which will be the goal of the article, and the capabilities of the communication network. The update of $\hat{x}$ at $t \in \mathcal{T}$ is modeled as $\hat{x}(t) = J_1 x(t^-) + J_2 \hat{x}(t^-)$ for all $t \in \mathcal{T}$, where J_1 and J_2 depend on the transmission delay. In between those triggering instants, a state estimator can be designed, dependent on the computational capabilities of the actuator, as $\dot{\hat{x}} = A_e \hat{x}$ for $t > t_0$ and $t \notin \mathcal{T}$. This formulation of A_e can represent for example a zero-order hold implementation at an actuator without computational capabilities, i.e., $A_e = 0$, or a closed loop model-based estimator in a smart actuator, i.e., $A_e = A + BK$. The triggering strategies derived in this paper assume that there are no computational capabilities and thus $A_e = 0$. Nevertheless, A_e will be included in this and the following section to show the generality of the model and stability conditions. Together with the continuity of the solution of (1), we can model the closed-loop system combined of system (1), controller (2), and the estimator with its reset condition as a discontinuous dynamical system (DDS) with state $\xi = \begin{bmatrix} x^\intercal \hat{x}^\intercal \end{bmatrix}^\intercal$ as

$$\begin{aligned} \dot{\xi}(t) &= \underbrace{\begin{bmatrix} A & BK \\ 0 & A_e \end{bmatrix}}_{A_\xi} \xi(t), \ \tau_{k-1} < t < \tau_k \quad \forall k \in \mathbb{N}, \\ \xi(t) &= \underbrace{\begin{bmatrix} I & 0 \\ J_1 & J_2 \end{bmatrix}}_{B_\xi} \xi(t^-), \ t \in \mathcal{T}, \end{aligned} \tag{4}$$

where $\xi(\tau_0 := t_0) = \begin{bmatrix} x_0^\intercal \ \hat{x}_0^\intercal \end{bmatrix}^\intercal =: \xi_0$. The goal is to use the structure of the DDS together with known stability results for such systems to derive conditions for event-triggered NCS that can be represented by this model. After deriving these results, we show explicitly how they can be used to derive static and dynamic triggering mechanisms. All the strategies that will be presented, aim for guaranteeing *exponential stability in the large* for the equilibrium at the origin of the closed-loop system.

Definition 1 (Definition 6.1.1 (j) in [20]) *The equilibrium at the origin of a dynamical system is exponentially stable in the large if there exist an $\alpha > 0$ and a $\gamma > 0$, and for any $\beta > 0$, there exists a $k > 0$, dependent on β, such that for all solutions ϕ of the dynamical system $\|\phi(t, t_0, \xi_0)\| \leq k\|\xi_0\|^\gamma \mathrm{e}^{-\alpha(t-t_0)}, \quad \forall t \geq t_0$, whenever $\|\xi_0\| < \beta$.*

2.2 Modeling triggering strategies and network

In the foregoing subsection, the elements in $\mathcal{T}$ were introduced as the time instants when current or delayed information is received at the actuator. Due to the assumption that the transmissions from the sensor to the actuator are imperfect, one can define other relevant sequences. In particular, we can define the sequence $(\tau_k^s)_{k\in\mathbb{N}}$, that contains all instants when information

is sent over the network. The corresponding set that contains all elements of $(\tau_k^s)_{k\in\mathbb{N}}$ is denoted as $\mathcal{T}^s$. The elements in $\mathcal{T}^s$ depend directly on the chosen triggering strategy. Throughout this article, we will deal with periodic event-triggered control (PETC) strategies, i.e., the trigger rule is checked only at evenly distributed time instants with a predefined period h. If the trigger rule is violated at $t = t_0 + kh =: t_k$ for some $k \in \mathbb{N}_0$, an event is triggered, i.e., the information $x(t_k)$ is sent over the network. Thus, the set $\mathcal{T}^s$ is given as

$$\begin{aligned}\mathcal{T}^s &= \{t \geq t_0 : t \in (t_k)_{k\in\mathbb{N}_0} \wedge \text{trigger rule violated at } t\} \\ &:= \{\tau_1^s, \tau_2^s, \dots\} \end{aligned} \tag{5}$$

without specifying a particular trigger rule yet.

We can further use the two sequences and corresponding sets to model the network imperfections. The considered imperfections in this article are either the loss of at most m consecutive packets that are sent over the network or the combination of at most m consecutive losses and a constant time delay $\delta = h$. Note that we will sometimes stick to using δ instead of h to show that the general model holds for general constant delays. Those two models can be interpreted as follows: the scenario with only packet losses being considered resembles the scenario where messages are dropped if they have a transmission time that is larger than a deadline that is much shorter than h and does not affect stability and performance of the control loop. The case with a constant delay $\delta = h$ and possible dropouts models the scenario where a delay of one sampling interval is acceptable by the control application and also integrated in the controller design. Thus, messages that arrive faster are buffered and messages that arrive later are discarded.

For the case of packet losses but no delay, the sequence of instants when current information is available at the actuator $(\tau_k)_{k\in\mathbb{N}}$ is a subsequence of $(\tau_k^s)_{k\in\mathbb{N}}$, such that at most m successive elements from $(\tau_k^s)_{k\in\mathbb{N}}$ are not contained in $(\tau_k)_{k\in\mathbb{N}}$. The fact that this information is undelayed, is represented by the update matrix $J := [J_1\ J_2] = [I\ 0]$. In the case of transmissions that are subject to delay and packet loss the scenario is modeled by $(\tau_k)_{k\in\mathbb{N}}$ being a subsequence of $(\tau_k^s + \delta)_{k\in\mathbb{N}}$ with at most m successive elements from $(\tau_k^s + \delta)_{k\in\mathbb{N}}$ being not contained in $(\tau_k)_{k\in\mathbb{N}}$. The presence of delayed information for the update at $\tau \in \mathcal{T}$ is then expressed by $J = [I\ 0]\mathrm{e}^{-A_\xi \delta}$.

An illustration of the different time sequences is given in Fig. 1. This also illustrates the corresponding index sets. From (5) one can see that for PETC with $\delta = 0$ or $\delta = h$, all elements of the sequences $(\tau_k)_{k\in\mathbb{N}}$ and $(\tau_k^s)_{k\in\mathbb{N}}$, are also elements of $(t_k)_{k\in\mathbb{N}_0}$. Thus one can define index sets that will be useful in the later analysis parts as

$$k \in \mathcal{I}^\tau (\text{resp. } \mathcal{I}^{\tau^s}) \Leftrightarrow t_k \in \mathcal{T} (\text{resp. } \mathcal{T}^s). \tag{6}$$

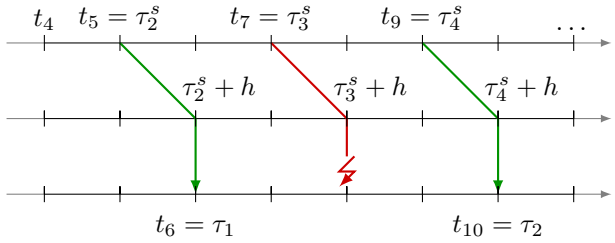

Fig. 1. Example to illustrate time sequences with delay.

As an example, the sequences in Fig. 1 show that $\{6, 10\} \in \mathcal{I}^\tau$ and $\{5, 7, 9\} \in \mathcal{I}^{\tau^s}$.

3 Non-monotonic stability results for event-triggered NCS

Before explicitly constructing trigger rules for the configurations described above, a sufficient stability condition is derived that fits the general model in (4) without assumptions on the triggering concept, the estimator, and the constant delays. The result is based on a known theorem for stability properties of general finite dimensional DDS, which is given in Appendix A. We use the structure of (4) to derive sufficient stability conditions that can be used to design triggering conditions for such systems. We do this in a two step approach, by providing a result that fits to the general model in (4), and deriving a corollary for a relevant subclass.

Proposition 2 *Observe the DDS given by* (4). *Assume that the unbounded discrete subset $\mathcal{T}$ of $\mathbb{R}_{\geq t_0}$ with no finite accumulation points satisfies*

$$0 < \underline{\eta} \leq \tau_{k+1} - \tau_k \leq \overline{\eta} \tag{7}$$

for all $k \in \mathbb{N}_0$ where $\tau_0 = t_0$. Assume that there exist a positive constant c and a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ such that, with $V(\xi) = \xi^\intercal P \xi$, for all $k \in \mathbb{N}$,

$$V(\phi(\tau_{k+1}, t_0, \xi_0)) - V(\phi(\tau_k, t_0, \xi_0)) \leq -c\|\phi(\tau_k, t_0, \xi_0)\|^2 \tag{8}$$

for all $\xi_0 \in \mathbb{R}^{2n}$ and all solutions ϕ of the DDS (4) *with $\phi(t_0, t_0, \xi_0) = \xi_0$. Then, the equilibrium $\xi_e = 0$ of* (4) *is exponentially stable in the large.*

PROOF. The proof is given in Appendix B. □

This proposition states that if one can show positive uniform upper and lower bounds on the actual inter-event intervals, i.e., the time between two instants when information is available at the actuator, and find a quadratic Lyapunov function that decreases in between those successful triggering instants in the sense of (8), one can

conclude exponential stability in the large of the equilibrium at the origin. The restriction to quadratic Lyapunov functions is motivated by the linear setup considered in this article. The general theorem in Appendix A could be used to derive more general results without restricting to quadratic Lyapunov functions. Due to the definition of $x, \hat{x}$, and ξ_0, the solution $\phi(\cdot, t_0, \xi_0)$ is given by $\phi(\cdot, t_0, \xi_0) = \begin{bmatrix} x^\intercal & \hat{x}^\intercal \end{bmatrix}^\intercal$. This insight is exploited in the following corollary that utilizes structural properties of (4). In particular, the zero blocks in B_ξ, that appear in the case without transmission delays, are crucial to reduce the stability analysis to a consideration of x instead of ξ under certain conditions.

Corollary 3 *Observe the DDS given by* (4)*, where* $\phi(\cdot, t_0, \xi_0) = \begin{bmatrix} x^\intercal & \hat{x}^\intercal \end{bmatrix}^\intercal$*. Assume*

$$\operatorname{rank} J_1 = n \quad \text{and} \quad J_2 = 0 \tag{9}$$

and that the unbounded discrete subset $\mathcal{T}$ *of* $\mathbb{R}_{\geq t_0}$ *with no finite accumulation points satisfies*

$$0 < \underline{\eta} \leq \tau_{k+1} - \tau_k \leq \overline{\eta} \tag{10}$$

for all $k \in \mathbb{N}_0$ *where* $\tau_0 = t_0$*. Assume that there exist a positive constant* c_x *and a positive definite matrix* $P_x \in \mathbb{R}^{n \times n}$ *such that for all* $k \in \mathbb{N}$*,*

$$x(\tau_{k+1})^\intercal P_x x(\tau_{k+1}) - x(\tau_k)^\intercal P_x x(\tau_k) \leq -c_x \|x(\tau_k)\|^2 \tag{11}$$

for all $\xi_0 = \begin{bmatrix} x_0^\intercal & \hat{x}_0^\intercal \end{bmatrix}^\intercal \in \mathbb{R}^{2n}$*. Then the equilibrium* $\xi_e = 0$ *of* (4) *is exponentially stable in the large.*

PROOF. We prove Corollary 3 using Proposition 2. Note that (7) is identical to (10).

$$P = \begin{bmatrix} P_x & 0 \\ 0 & \left(J_1^{-1}\right)^\intercal P_x J_1^{-1} \end{bmatrix}, \tag{12}$$

where existence of J_1^{-1} is guaranteed when J_1 has full rank. Furthermore, from $\operatorname{rank} J_1^{-1} = n$ and $P_x \succ 0$ one concludes $\left(J_1^{-1}\right)^\intercal P_x J_1^{-1} \succ 0$ and thus $P \succ 0$. To prove exponential stability in the large, one computes

$$\begin{aligned}
&V(\phi(\tau_{k+1}, t_0, \xi_0)) - V(\phi(\tau_k, t_0, \xi_0)) \\
&= 2(x(\tau_{k+1}^-)^\intercal P_x x(\tau_{k+1}^-) - x(\tau_k^-)^\intercal P_x x(\tau_k^-)) \\
&= 2(x(\tau_{k+1})^\intercal P_x x(\tau_{k+1}) - x(\tau_k)^\intercal P_x x(\tau_k)) \\
&\leq -2c_x \|x(\tau_k)\|^2 \leq -\underbrace{2c_x \frac{1}{1 + \|J_1\|^2}}_{:=c} \left\| \begin{bmatrix} x(\tau_k) \\ J_1 x(\tau_k) \end{bmatrix} \right\|^2 \\
&= -c \left\| \begin{bmatrix} x(\tau_k)^\intercal & \hat{x}(\tau_k)^\intercal \end{bmatrix}^\intercal \right\|^2 = -c \|\phi(\tau_k, t_0, \xi_0)\|^2
\end{aligned} \tag{13}$$

for all $k \in \mathbb{N}$, $\xi_0 \in \mathbb{R}^{2n}$, where the second equality uses $\hat{x}(\tau_{k+1}) = J_1 x(\tau_{k+1}^-)$ since $J_2 = 0$, and one uses the fact that $\left\| \begin{bmatrix} x(\tau_k)^\intercal & (J_1 x(\tau_k))^\intercal \end{bmatrix}^\intercal \right\|^2 \leq (1 + \|J_1\|^2)\|x(\tau_k)\|^2$. This shows that (8) holds and thus all assumptions for Proposition 2 are satisfied and exponential stability in the large is shown. □

Remark 4 *The delay free case is modeled as* $J_1 = I$ *and* $J_2 = 0$ *and thus, the additional assumptions of Corollary 3 are always satisfied.*

Remark 5 *Note that* $\tau_0 = t_0$ *and* τ_1 *is the first successful transmission time with* $\tau_1 > \tau_0$*. Thus, the results assume no received information at* t_0*. Note furthermore that the decrease condition has to be satisfied only between successful triggering times, not between* t_0 *and* τ_1*.*

In the following sections, we will use Proposition 2 and Corollary 3 to derive triggering strategies and to prove the results, i.e., we will always guarantee the desired bounds on the inter-event intervals and analyze the evolution of a Lyapunov function at time instants when information is successfully received.

4 Static state-dependent trigger rule

In this section, the potential of the stability results presented above for designing static, state-dependent trigger rules is shown. The considered triggering strategy, as described in Section 2, is a PETC strategy and the actuation strategy is of zero order hold type, i.e., $A_e = 0$. Note that for this case, it can be beneficial to compute the control input at the sensor and transmit it instead of the state, but throughout the article we will address the general setup which allows to transmit the whole state. The considered network model for the static case is the setup with at most m successive packet losses and no delay. The ideas are mainly based on the derivations in [17]. The major difference is, that compared to the approach in [17], no additional trigger rule is necessary.

The role of the periodic sequence $(t_k)_{k \in \mathbb{N}_0}$ for PETC was already explained in Subsection 2.2. To further use this in the upcoming derivation, we introduce the discrete-time system

$$x_{k+1} = A_d x_k + B_d u_k, \quad x_0 = x(t_0) \tag{14}$$

which is, with $A_d = \mathrm{e}^{Ah}$, $B_d = \int_0^h \mathrm{e}^{As} \mathrm{d}s B$, and $u_k = K\hat{x}(t_k)$ an exact discretization of (1) with $x_k = x(t_k), k \in \mathbb{N}_0$. Analogously denote $\hat{x}_k := \hat{x}(t_k)$. Throughout the article, assume that h is a so called non-pathological sampling time, i.e., the discrete-time system (14) is stabilizable, provided that (A, B) is stabilizable. Corresponding conditions on h can be found for example in [24], Section 3.4.

In the following subsections, a controller, a corresponding Lyapunov function and a trigger rule will be derived such that one can show uniform boundedness of the inter-event intervals and a guaranteed decrease of the Lyapunov function between successful transmission instants. Since we analyze the delay free case, this will be used to show exponential stability in the large using Corollary 3.

4.1 Controller and trigger rule

To begin with, the trigger error $e(t) = \hat{x}(t) - x(t)$ for $t \geq t_0$ and its discrete-time counterpart

$$e_k := e(t_k) = \hat{x}(t_k) - x(t_k) = \hat{x}_k - x_k,\ k \in \mathbb{N}_0 \quad (15)$$

are defined. Using the trigger error e_k and the fact that $A_e = 0$, one can compute a closed-loop representation of the discretized system (14) for all $k \in \mathbb{N}_0$ as

$$x_{k+1} = \underbrace{(A_d + B_d K)}_{A_1} x_k + \underbrace{B_d K}_{B_1} e_k \quad (16)$$

$$e_{k+1} = \begin{cases} 0, & \text{if } k+1 \in \mathcal{I}^\tau \\ \underbrace{(I - B_d K)}_{A_2} e_k + \underbrace{(I - (A_d + B_d K))}_{B_2} x_k, & \text{else} \end{cases}$$

As mentioned in the beginning of the section, it is assumed that (A_d, B_d) is stabilizable. Thus, one can compute a controller K_x and a corresponding $P_x \succ 0$ s.t.

$$x_{k+1}^\intercal P_x x_{k+1} - x_k^\intercal P_x x_k \leq -\theta^2 \|x_k\|^2 + \|e_k\|^2 \quad (17)$$

with $\theta > 0$ holds for all $k \in \mathbb{N}_0$, when $K = K_x$, which we assume in the remainder of this section. Note also that

$$\lambda_{\min}(P_x)\|x_k\|^2 \leq x_k^\intercal P_x x_k \leq \lambda_{\max}(P_x)\|x_k\|^2 \quad (18)$$

holds. In view of this standard derivation, a trigger rule where triggering is necessary at $t \in (t_k)_{k\in\mathbb{N}}$ if

$$\|e_k\| > \sigma \|x_k\|, \quad (19)$$

with $\sigma > 0$, is used. In case of no packet dropouts, the condition $\sigma < \theta$ is known to be sufficient for global exponential stability, cf. [8], [12]. We will compute a new bound on σ in the following for the case of m successive packet losses. Therefore, we aim for using P_x in a quadratic Lyapunov function candidate for Corollary 3. Thus, it is to be shown that the actual inter-event intervals are uniformly bounded to satisfy (10) as well as that the evolution of the Lyapunov function between successful triggering instants satisfies (11). This will be the goal of the next two subsections. Before we start those derivations, we will give the following lemma that provides useful bounds for the further analysis.

Lemma 6 *Assume that the trigger rule* (19) *with* $0 < \sigma < 1$ *is not violated at some time* $t = t_l$*, i.e.,* $\|e_l\| \leq \sigma\|x_l\|$ *and thus* $l \notin \mathcal{I}^{\tau^s}$*. Assume furthermore that the last successful triggering instant prior to* t_l *occurred at some time* $t = t_i$*, i.e., there does not exist a* $j \in \mathcal{I}^\tau$ *such that* $i < j < l$*. Then the following inequalities hold.*

(1) $\|e_l\| \geq \|x_i\| - \|x_l\|$
(2) $\|x_l\| \geq \frac{1}{1+\sigma}\|x_i\|$
(3) $\|x_l\| \leq \frac{1}{1-\sigma}\|x_i\|$

PROOF.

(1) From the definition of the trigger error in (15), one knows that $\hat{x}_l = x_i = x_l + e_l$ and thus $\|x_i\| \leq \|x_l\| + \|e_l\|$, which directly gives the first inequality.
(2) The first inequality is equivalent to $\|x_l\| \geq \|x_i\| - \|e_l\|$. Thus, since the trigger rule is not violated at t_l, it follows that $\|x_l\| \geq \|x_i\| - \sigma\|x_l\|$ which is equivalent to the second inequality.
(3) Again, from the definition of the trigger error in (15), it follows that $x_l = x_i - e_l$ and thus $\|x_l\| \leq \|x_i\| + \|e_l\|$. As in the derivation of the second inequality, the trigger rule can be used to show that $\|x_l\| \leq \|x_i\| + \sigma\|x_l\|$, which is equivalent to the third inequality for $\sigma < 1$. □

4.2 Bounds on actual inter-event intervals

It is clear that in a PETC setup, the inter-event intervals are lower bounded by the sampling time h. In the next result, we will state and prove that the actual inter-event intervals, i.e., the time between two successful triggering instants, are also uniformly upper bounded.

Lemma 7 *Assume system* (1) *is controlled with* (2)*, where* $K = K_x$*. Furthermore, assume that transmissions are demanded according to the PETC strategy with trigger rule* (19) *where* $0 < \sigma < \min\{\theta, 1\}$*. If at most* m *successive transmissions can be lost, the interval between two successful triggering instants is uniformly upper bounded.*

PROOF. For the proof of this technical result, we refer to the proof of Lemma 2 in the conference paper [17].

Remark 8 *Using the same calculation as for Lemma 7, one can show that* $\tau_1 - t_0$ *is upper bounded as well and thus* (10) *is satisfied for all* $k \in \mathbb{N}_0$*.*

4.3 Decrease of the Lyapunov function

In this subsection, the evolution of $x_k^\intercal P_x x_k$ will be analyzed between successful triggering instants. The goal is to derive a bound on σ, such that condition (11) in

Corollary 3 is satisfied in case of at most m successive packet losses.

Before we start the derivation, we reconsider the closed-loop representation in (16) that can be rewritten for $k+1 \notin \mathcal{I}^\tau$ as

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = F \begin{bmatrix} x_k \\ e_k \end{bmatrix} \text{ with } F := \begin{bmatrix} A_1 & B_1 \\ B_2 & A_2 \end{bmatrix}. \tag{20}$$

Using this, one can derive an explicit formula for the evolution of x_k and e_k between two consecutive successful triggering instants. Assume $j \geq k : j \notin \mathcal{I}^\tau$ and that there does not exist an i such that $i \in \mathcal{I}^\tau$ and $t_k < t_i < t_j$. Then, $\begin{bmatrix} x_j^\intercal & e_j^\intercal \end{bmatrix}^\intercal = F^{j-k} \begin{bmatrix} x_k^\intercal & e_k^\intercal \end{bmatrix}^\intercal$ and we denote the $n \times n$ blocks in F^{j-k} according to

$$F^{j-k} =: \begin{bmatrix} A_{1|j-k} & B_{1|j-k} \\ B_{2|j-k} & A_{2|j-k} \end{bmatrix}. \tag{21}$$

Using this notation, the evolution of $x_k^\intercal P_x x_k$ can now be analyzed. Two cases need to be investigated. The first case is that the trigger rule is violated at every time instant between two successive successful triggering instants τ_k and τ_{k+1}. In this case the chosen value for σ cannot be exploited to bound the evolution of the Lyapunov function but the condition

$$A_{1|\tilde{m}+1}^\intercal P_x A_{1|\tilde{m}+1} - P_x \preceq -rI \tag{22}$$

for $\tilde{m} \in \{1, \dots, m\}$ and $r > 0$ is sufficient for guaranteeing that (11) is satisfied with $c_x = r$ according to (21) and the fact that $e(\tau_k) = 0$. Note that although (22) does not depend on σ, it is not an actual necessary condition for exponential stability in the large, since we fixed P_x in the quadratic Lyapunov function, i.e., the one that we obtain from the controller design part.

For the second case, that remains to be studied, denote $\tau_k = t_{i_1}$ and $\tau_{k+1} = t_{i_2}$ and use the dissipation inequality (17) that directly translates to

$$x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq \sum_{k \in [i_1, i_2)} -\theta^2 \|x_k\|^2 + \|e_k\|^2. \tag{23}$$

Using the definition of the trigger rule (19), the index sets (6), and the fact that $e_{i_1} = 0$, one can conclude

$$\begin{aligned} x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq -(\theta^2 - \sigma^2) \sum_{k \in (i_1, i_2) \setminus \mathcal{I}^{\tau^s}} \|x_k\|^2 \\ -\theta^2 \|x_{i_1}\|^2 + \sum_{k \in (i_1, i_2) \cap \mathcal{I}^{\tau^s} \setminus \mathcal{I}^\tau} -\theta^2 \|x_k\|^2 + \|e_k\|^2, \end{aligned} \tag{24}$$

where the first sum represents time instants when no triggering is necessary, i.e., $\|e_k\|^2 \leq \sigma^2 \|x_k\|^2$ holds in this case, and the second sum represents time instants when triggering is necessary but not successful.

In the remainder of this subsection, the goal is to bound the two sums in (24) dependent on $\|x_{i_1}\|^2$ to guarantee that $x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq -r\|x_{i_1}\|^2$ for some $r > 0$. As discussed above, we can assume that there exists at least one time instant, but not necessarily more, when no triggering is necessary between τ_k and τ_{k+1}. Thus, using inequality (2) of Lemma 6 and only one term of the first sum it follows that

$$\begin{aligned} x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq -(\theta^2 - \sigma^2) \left(\frac{1}{1+\sigma} \right)^2 \|x_{i_1}\|^2 \\ -\theta^2 \|x_{i_1}\|^2 + \sum_{k \in (i_1, i_2) \cap \mathcal{I}^{\tau^s} \setminus \mathcal{I}^\tau} -\theta^2 \|x_k\|^2 + \|e_k\|^2. \end{aligned} \tag{25}$$

The next goal is to bound the norm of the trigger error for time instants when triggering is necessary, i.e., in the last sum in (25). For this purpose we exemplarily consider a time instant $t_j \in [t_{i_1}, t_{i_2})$ when triggering is either successful or not necessary followed by $\tilde{m}$ time instants when triggering is necessary but not successful. Then using (21), one can compute that

$$\begin{aligned} \|e_{j+\tilde{m}}\| &= \|A_{2|\tilde{m}} e_j + B_{2|\tilde{m}} x_j\| \\ &\leq \left(\|A_{2|\tilde{m}}\| \sigma + \|B_{2|\tilde{m}}\| \right) \|x_j\| \\ &\leq \left(\|A_{2|\tilde{m}}\| \sigma + \|B_{2|\tilde{m}}\| \right) \frac{1}{1-\sigma} \|x_{i_1}\| \end{aligned} \tag{26}$$

where we used inequality (3) of Lemma 6 for the last inequality. Thus, it holds that

$$\begin{aligned} \|e_{j+\tilde{m}}\|^2 &\leq \left(\left(\|A_{2|\tilde{m}}\| \sigma + \|B_{2|\tilde{m}}\| \right) \frac{1}{1-\sigma} \right)^2 \|x_{i_1}\|^2 \\ &=: c_{\tilde{m}}(\sigma) \|x_{i_1}\|^2. \end{aligned} \tag{27}$$

Directly by the assumption of at most m consecutive transmissions that are unsuccessful, it is known that the sum in (25) consists of at most m terms. However, one can not directly use the term $\sum_{k=1}^m c_k(\sigma) \|x_{i_1}\|$ to bound the sum in (25), since it is not guaranteed that there are m consecutive time instants when a transmission is necessary and fails. It could also be the case that a time instant when a transmission is necessary but unsuccessful is followed by a time instant when no transmission is necessary, since there is no explicit demand for retransmissions. To represent the worst case over all those possibilities, we introduce

$$\bar{c}(m, \sigma) = \max\{c_1(\sigma) + \dots + c_m(\sigma), \dots, m c_1(\sigma)\}.$$

The notation in the brackets indicates two of the possible cases. The first one $(c_1 + \dots + c_m)$ describes the error term for the case when at m successive time instants triggering is necessary but not successful. The second

one (mc_1) describes it for the possible case that always after a unsuccessful transmission no transmission is necessary at the next time instant, which is possible at most m times. The worst case over all of the possible combinations in above sense can be computed by the following integer linear program (ILP)

$$\begin{aligned} \bar{c}(m,\sigma) = \underset{x}{\text{maximize}} \quad & \begin{bmatrix} c_1(\sigma) & \dots & c_m(\sigma) \end{bmatrix} x \\ \text{subject to} \quad & x \in \mathbb{N}_0^m, \\ & x_i \geq x_j \text{ for } i < j, \\ & \sum_{i=1}^{m} x_i = m. \end{aligned} \tag{28}$$

Thus, inequality (25) can be simplified to

$$\begin{aligned} & x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq -\frac{\theta^2 - \sigma^2}{(1+\sigma)^2} \|x_{i_1}\|^2 \\ & -\theta^2 \|x_{i_1}\|^2 + \bar{c}(m,\sigma)\|x_{i_1}\|^2 - \sum_{k \in (i_1,i_2) \cap \mathcal{I}^{\tau^s} \setminus \mathcal{I}^\tau} \theta^2 \|x_k\|^2. \end{aligned} \tag{29}$$

The remaining sum could simply be treated by the nonnegativity of the norm. Since this introduces conservativity, we aim for a tighter bound. Assume there exist $k \in (i_1, i_2) \cap \mathcal{I}^{\tau^s} \setminus \mathcal{I}^\tau$ and $r > 0$ such that $x_k^\intercal P_x x_k - x_{i_1}^\intercal P_x x_{i_1} \leq -r\|x_{i_1}\|^2 - \bar{c}(m,\sigma)\|x_{i_1}\|^2$. Then, condition (11) in Corollary 3, which is the desired decrease of the Lyapunov function that needs to be guaranteed by the trigger rule, is directly satisfied with $c_x = r$. If the assumption is not satisfied, one knows that for all $k \in (i_1, i_2) \cap \mathcal{I}^{\tau^s} \setminus \mathcal{I}^\tau$ and $r > 0$, it holds that $x_k^\intercal P_x x_k - x_{i_1}^\intercal P_x x_{i_1} > -(r + \bar{c}(m,\sigma))\|x_{i_1}\|^2$ and thus, due to (18) and nonnegativity of the norm, we have

$$\|x_k\|^2 \geq \max\left\{0, \frac{\lambda_{\min}(P_x) - r - \bar{c}(m,\sigma)}{\lambda_{\max}(P_x)}\right\} \|x_{i_1}\|^2 . \tag{30}$$

Inserting this inequality in (29) results in

$$\begin{aligned} & x_{i_2}^\intercal P_x x_{i_2} - x_{i_1}^\intercal P_x x_{i_1} \leq -\Bigg(\frac{\theta^2 - \sigma^2}{(1+\sigma)^2} + \theta^2 - \bar{c}(m,\sigma) \\ & +m\theta^2 \max\left\{0, \frac{\lambda_{\min}(P_x) - r - \bar{c}(m,\sigma)}{\lambda_{\max}(P_x)}\right\}\Bigg) \|x_{i_1}\|^2. \end{aligned} \tag{31}$$

Using this derivation, we are now able to state a lemma that poses conditions on σ to guarantee a decrease of the Lyapunov function between every two successful triggering instants.

Lemma 9 *Assume P_x, θ, and $K = K_x$ being chosen such that* (17) *holds. Assume furthermore that one can find $r \in (0, \theta^2)$ such that* (22) *is satisfied for all $\tilde{m} \in \{1, \dots, m\}$ and that there exists $\sigma : 0 < \sigma < \min\{1, \theta\}$ such that*

$$\begin{aligned} & r - \Bigg(\theta^2 + \frac{\theta^2 - \sigma^2}{(1+\sigma)^2} - \bar{c}(\tilde{m},\sigma) \\ & +\tilde{m}\theta^2 \max\left\{0, \frac{\lambda_{\min}(P_x) - r - \bar{c}(\tilde{m},\sigma)}{\lambda_{\max}(P_x)}\right\}\Bigg) \leq 0 \end{aligned} \tag{32}$$

with $\bar{c}(\tilde{m},\sigma)$ as in (28) *and $c_i(\sigma)$ as in* (27) *for all $\tilde{m} \in \{1, \dots, m\}$ as well. Then, the bound*

$$x(\tau_{k+1})^\intercal P_x x(\tau_{k+1}) - x(\tau_k)^\intercal P_x x(\tau_k) \leq -r\|x(\tau_k)\|^2 \tag{33}$$

holds for all $k \in \mathbb{N}$ if triggering is demanded according to the trigger rule (19) *and at most m consecutive transmissions are lost.*

PROOF. The proof follows directly from the derivations in this subsection and the restriction $r < \theta^2$ covers the case that all transmissions are successful. □

Note that the bound on successive packet losses m influences the condition of Lemma 9 in the sense that (22) and (32) need to hold for all $\tilde{m} \in \{1, \dots, m\}$. We can now combine Lemma 7 and Lemma 9 to state the main result for the static triggering case.

Theorem 10 *Assume that system* (1) *is controlled with controller* (2)*, where $K = K_x$. Assume furthermore that transmissions are necessary if the PETC trigger rule* (19)*, where σ satisfies the conditions of Lemma 9, is violated and that the network guarantees that at most m successive packets are lost and the delay is considered as $\delta = 0$. Then the equilibrium at $\xi = 0$ of the DDS* (4) *is exponentially stable in the large.*

PROOF. Since the delay free case is investigated, the elements of the update matrix J are given as $J_1 = I$ and $J_2 = 0$. Thus, condition (9) is satisfied. Furthermore from Lemma 7, Remark 8, and the properties of the PETC scenario it can be seen that (10) is satisfied for all $k \in \mathbb{N}_0$. Since σ is chosen according to the conditions of Lemma 9, it is also guaranteed that (11) holds for all $k \in \mathbb{N}$ with $c_x = r$. Thus, by Corollary 3, the equilibrium at $\xi = 0$ of the DDS (4) is exponentially stable in the large. □

Remark 11 *Note that in [17], an additional trigger rule was necessary to guarantee the desired decrease of the Lyapunov function. This was mainly since only one possible combination of $c_i(\sigma)$ was considered and not the worst case as it is done here using an ILP. The price to pay is that the ILP needs to be solved for all values of σ for which one wants to check feasibility of* (32)*. Since this is an offline computation, it can be seen as a tradeoff between offline computation to find the largest possible*

value for σ, online computation as in the approach with the additional trigger rule, and communication frequency since this can be reduced with a large value of σ.

Remark 12 *Note that it is in principle possible to translate the ideas to the delay case. Nevertheless, there are certain issues. For example, it is hard to come up with a counterpart to all inequalities in Lemma 6. Furthermore, the results seem to be very conservative such that often it is not possible to compute a positive threshold σ. For this reason, the delay case will not be presented here, but only in the following section, where it is possible to reduce the overall conservatism.*

5 Dynamic triggering strategies

In the previous section, we derived a static trigger rule such that exponential stability in the large can be guaranteed, although unsuccessful transmissions are possible. While such static trigger rules have benefits in terms of computation and implementation, they naturally imply a certain conservativity. This can in particular be noticed when trying to extend the previous approach to a scenario with delays, as mentioned in Remark 12. Therefore, this section focuses on a PETC mechanism that can be seen as a dynamic triggering mechanism. The idea is to guarantee directly the satisfaction of the conditions in Proposition 2, respectively Corollary 3, for the case with, respectively without, delay. The setup being considered is again given by an actuator without computational capabilities, i.e., $A_e = 0$ and a network with at most m consecutive packet losses and either no delay or the constant delay $\delta = h$.

5.1 Dynamic triggering without delay

Assume that A_d, B_d denote the discretized system matrices as in the previous section and a stabilizing controller K for (14) is given. To use Corollary 3 for guaranteeing exponential stability in the large, the triggering mechanism stores the time instant when the last successful transmission over the network occurred, and the corresponding value of the state. It checks whether the time since the last received transmission exceeds a desired threshold νh with $\nu \in \mathbb{N}$. If this is the case, the state information is sent over the network to guarantee existence of a uniform upper bound on the actual inter-event intervals ($\bar{\eta} \leq (\nu + m)h$) as demanded by (10).

Further derivations aim at guaranteeing that condition (11) in Corollary 3 holds. The first step, that can be done offline, is to find a suitable Lyapunov function candidate, i.e., a positive definite matrix $\bar{P}_x$ that takes the role of P_x in (11). We compute the candidate, if possible, by the set of LMIs

$$A_{1|j}^{\intercal} \bar{P}_x A_{1|j} - (1 + \bar{r})\bar{P}_x \preceq 0, \quad j \in \{1, \ldots, m+1\} \quad (34)$$

for a fixed design parameter $\bar{r} > 0$ that can be solved for a sufficiently small $\bar{r}$ if there exists a common quadratic Lyapunov function for a switched system $\tilde{x}_{k+1} = A_{1|j}\tilde{x}_k$ with j as in (34), see [16]. The condition (34) is motivated by the fact that if one can find a $\bar{P}_x \succ 0$ satisfying (34), one knows according to the definition of $A_{1|j}$ in Subsection 4.3 and the fact that $e_k = 0$ for $k \in \mathcal{I}^{\mathcal{T}}$ that for $j \in \{1, \ldots, m+1\}$,

$$x_{k+j}^{\intercal} \bar{P}_x x_{k+j} - x_k^{\intercal} \bar{P}_x x_k \leq -\bar{r}\lambda_{\min}(\bar{P}_x)\|x_k\|^2, \quad (35)$$

if $k \in \mathcal{I}^{\tau}$ and $k + j \notin \mathcal{I}^{\tau}$ for $j \in \{1, \ldots, m\}$. Thus, in the scenario where a successful transmission is immediately followed by m time instants without successful transmission and a successful transmission afterwards, one knows that (11) is satisfied with $c_x = \bar{r}\lambda_{\min}(\bar{P}_x)$.

The last and main part of the mechanism and the most significant novelty compared to Section 4, is to guarantee that (11) holds with some desired value $\bar{c}_x > 0$ for all other cases. In Section 4, we enforced this guarantee by deriving a static trigger rule. Now, a dynamic mechanism relying on predictions is derived. Therefore, at all periodic time instants $t_k, k \in \mathbb{N}$, when the triggering condition is checked, one computes a predicted value for the state x at time $t_{k+m-\bar{m}_k+1}$, where $\bar{m}_k$ denotes the number of unsuccessful transmissions between the last successful transmission and time $t = t_k$. If we call this predicted value x_k^p and the value of the state at the last successful transmission x_k^{ref}, the current state is sent over the network if

$$(x_k^p)^{\intercal} \bar{P}_x x_k^p - (x_k^{\text{ref}})^{\intercal} \bar{P}_x x_k^{\text{ref}} > -\bar{c}_x \|x_k^{\text{ref}}\|^2. \quad (36)$$

Note that the prediction can be computed using the matrix F in (21), as $x_k^p = \begin{bmatrix} I & 0 \end{bmatrix} F^{1+m-\bar{m}_k} \begin{bmatrix} x_k^{\intercal} & e_k^{\intercal} \end{bmatrix}^{\intercal}$. The whole triggering mechanism is summarized in Algorithm 1. Note that it is directly influenced by m, since for a larger value of m it is harder to find $\bar{P}_x$ satisfying (34) and the prediction horizon for x_k^p in (36) gets longer.

Theorem 13 *Assume that system* (1) *is controlled with controller* (2) *with a given gain K. Assume furthermore that necessary transmissions are detected according to the PETC mechanism specified by Algorithm 1 and that the network guarantees that at most m successive packets are lost and the delay is considered as $\delta = 0$. If there exists a positive definite matrix $\bar{P}_x$ satisfying* (34) *with $\bar{r} > 0$, and the parameters of the triggering mechanism are chosen as $\nu \in \mathbb{N}, \bar{c}_x > 0$, then the equilibrium at $\xi = 0$ of the DDS* (4) *is exponentially stable in the large.*

PROOF. The setup can be described by the DDS (4) with $J_1 = I$ and $J_2 = 0$. Thus, one can use Corollary 3 to prove the result. The boundedness condition (10) is directly satisfied by the periodic event detection, the

Algorithm 1 Triggering mechanism with $\delta = 0$.

if $t = t_0$ **then**
  $i^{\text{ref}} \leftarrow 0$, $x^{\text{ref}} \leftarrow \hat{x}_0$, $\bar{m} \leftarrow 0$
  $V^{\text{ref}} \leftarrow \left(x^{\text{ref}}\right)^\intercal \bar{P}_x x^{\text{ref}}$ with $\bar{P}_x$ s.t. (34) holds
else if $t \in (t_k)_{k\in\mathbb{N}}$ **then**
  $x^p \leftarrow \begin{bmatrix} I & 0 \end{bmatrix} F^{1+m-\bar{m}} \begin{bmatrix} x_k \\ e_k \end{bmatrix}$, $V^p \leftarrow (x^p)^\intercal \bar{P}_x x^p$
  if $(k - i^{\text{ref}} \geq \nu)$ OR $(V^p - V^{\text{ref}} > -\bar{c}_x \|x^{\text{ref}}\|^2$ as in (36)) **then**
    send x_k over the network and wait for ACK
    if transmission is successful **then**
      $i^{\text{ref}} \leftarrow k$, $V^{\text{ref}} \leftarrow x_k^\intercal \bar{P}_x x_k$, $\bar{m} \leftarrow 0$
    else
      $\bar{m} \leftarrow \bar{m} + 1$
  else
    no transmission of x_k necessary

bounded number of successive packet losses, and the timeout condition. Thus (10) holds with $\underline{\eta} \geq h$ and $\overline{\eta} \leq (\nu + m)h$.

It remains to show that (11) holds with $P_x = \bar{P}_x$ and some $c_x > 0$ for all τ_k with $k \in \mathbb{N}$. This is shown in an iterative fashion. One knows by definition that there is a successful transmission at time $t = \tau_k$ and assumes without loss of generality that $\tau_k = t_i$ for some $i \in \mathbb{N}$. If the next successful transmission occurs after at most $m+1$ time instants, i.e., $\tau_{k+1} \in \{t_{i+1}, \dots, t_{i+m+1}\}$, one knows from (34) that (11) holds with $c_x = \bar{r}\lambda_{\min}(\bar{P}_x)$.

Before we analyze the remaining cases, observe that x_k^p is defined as a prediction over $m+1-\bar{m}_k$ time steps. Assume now that a transmission is initiated at t_k but the packet is lost. Then, $\bar{m}_{k+1} = \bar{m}_k + 1$ and thus x_{k+1}^p is a prediction over $m+1-\bar{m}_{k+1} = m-\bar{m}_k$ time steps. Thus, it is always guaranteed that $x_{k+1}^p = x_k^p$ if triggering is necessary at time t_k but the packet is lost. Thus, for the case that $\tau_{k+1} \notin \{t_{i+1}, \dots, t_{i+m+1}\}$, one knows that the triggering condition cannot be violated at time t_{i+1} since it would be violated at all successive time instants as well and successfully sent at latest at t_{i+m+1}. Due to the fact that $x_{i+1}^p = x_{i+1+(1+m)}$ and the fact that triggering is not necessary at t_{i+1}, the triggering mechanism guarantees that (11) holds with $c_x = \bar{c}_x$ if $\tau_{k+1} = t_{i+m+2}$. One proceeds in an iterative fashion. If $\tau_{k+1} > t_{i+m+2}$, one knows that no triggering was necessary at t_{i+2} and thus according to the trigger rule, (11) holds with $c_x = \bar{c}_x$ if $\tau_{k+1} = t_{i+m+3}$. Thus, one can conclude that (11) holds for all $k \in \mathbb{N}$ with $c_x = \min\{\bar{c}_x, \bar{r}\lambda_{\min}(\bar{P}_x)\}$ and the proof is completed. □

Remark 14 *Note that we used the fact that we can predict the state with exact precision in the derivation of the triggering mechanism. This is only possible in a linear scenario without disturbances. If disturbances were present, one had to use for example a worst case prediction that exploits bounds on disturbances.*

Remark 15 *Note that the triggering mechanism relies, as the static mechanism, on acknowledgement messages in the network. They are used to determine the last successfully transmitted state, the number of packet losses since the last successful transmission, and to infer the error between the current and the last successfully transmitted state. An ACK for a successful transmission at time t_k must arrive at the sensor at latest at $t_{\overline{k+1}}$.*

Next, a further remark is given. This additional remark discusses, how the dynamic triggering mechanism can be implemented in a computationally efficient manner.

Remark 16 *For the case $m = 0$, it holds that $x_k^p = \begin{bmatrix} I & 0 \end{bmatrix} F \begin{bmatrix} x_k^\intercal & \hat{x}_k^\intercal - x_k^\intercal \end{bmatrix}^\intercal$ for all $k \in \mathbb{N}_0$. Thus, (36) can be written as a static quadratic condition, as e.g. considered in standard PETC without loss [12], given by $\xi_k^\intercal Q_0 \xi_k > 0$ with*

$$Q_i = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \tilde{F}_i^\intercal \begin{bmatrix} \bar{P}_x & 0 \\ 0 & 0 \end{bmatrix} \tilde{F}_i \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\bar{P}_x + c_x I \end{bmatrix}$$

where $\tilde{F}_i = F^{1+i}$. For the case $m > 0$, it can be similarly seen that, depending on $\bar{m}_k$, one needs to choose and evaluate online only one condition from the set of $m+1$ quadratic trigger rules $\xi_k^\intercal Q_{m-\bar{m}_k} \xi_k > 0$ to check (36).

5.2 Dynamic triggering with delay

This subsection is concerned with the same setup as before, except for the presence of a constant transmission delay $\delta = h$. The main difference compared to the previous subsection is that Corollary 3 cannot be applied anymore, since J does not satisfy the conditions of Corollary 3 with $\delta = h$. Instead, the derivations in this subsection have to focus on the satisfaction of the conditions in Proposition 2. In this proposition, the solution of (4) for $t \in \mathcal{T}$ plays an important role. According to (4) and the fact that $\hat{x}(\tau_k) = x(\tau_k - h)$ for $\delta = h$, the solution evaluated at $t \in \mathcal{T}$ is given as $\xi(\tau_k) = \begin{bmatrix} x(\tau_k)^\intercal & x(\tau_k - h)^\intercal \end{bmatrix}^\intercal$. According to the notation that was introduced in (14) and the index sets introduced in (6), this states that $\xi_k := \begin{bmatrix} x_k^\intercal & \hat{x}_k^\intercal \end{bmatrix}^\intercal = \begin{bmatrix} x_k^\intercal & x_{k-1}^\intercal \end{bmatrix}^\intercal$ if $k \in \mathcal{I}^{\mathcal{T}}$. Since $\xi_k = \begin{bmatrix} x_k^\intercal & x_{k-1}^\intercal \end{bmatrix}^\intercal$ only holds for $k \in \mathcal{I}^{\mathcal{T}}$, we introduce the auxiliary state vector $\tilde{\xi}_k = \begin{bmatrix} x_k^\intercal & x_{k-1}^\intercal \end{bmatrix}^\intercal$ for all $k \in \mathbb{N}_0$ with some initial condition $x_{-1} \in \mathbb{R}^n$ and analyze its evolution. Therefore, we introduce a new definition for a trigger error, as a counterpart to (15), as $e_k^\delta = \hat{x}_k - x_{k-1}$, for $k \in \mathbb{N}_0$. Based on that definition and $A_e = 0$, a closed-loop representation for the case $\delta = h$ can be given for

all $k \in \mathbb{N}_0$ by

$$\tilde{\xi}_{k+1} = \underbrace{\begin{bmatrix} A_d & B_d K \\ I & 0 \end{bmatrix}}_{A_1^\delta} \tilde{\xi}_k + \underbrace{\begin{bmatrix} B_d K \\ 0 \end{bmatrix}}_{B_1^\delta} e_k^\delta,$$

$$e_{k+1}^\delta = \begin{cases} 0, & \text{if } k+1 \in \mathcal{I}^\tau \\ \underbrace{I}_{A_2^\delta} e_k^\delta + \underbrace{\begin{bmatrix} -I & I \end{bmatrix}}_{B_2^\delta} \tilde{\xi}_k, & \text{otherwise} \end{cases} \tag{37}$$

as a counterpart to (16). Since, in order to come up with an analogous approach to Section 5.1 we will employ predictions over a given horizon, the next step is to provide a counterpart to the F matrix in the delay free case, i.e., to (21). To this end, we rewrite the closed-loop equation (37) for $k+1 \notin \mathcal{I}^\mathcal{T}$ as

$$\begin{bmatrix} \tilde{\xi}_{k+1} \\ e_{k+1}^\delta \end{bmatrix} = \underbrace{\begin{bmatrix} A_1^\delta & B_1^\delta \\ B_2^\delta & A_2^\delta \end{bmatrix}}_{=:F_\delta} \begin{bmatrix} \tilde{\xi}_k \\ e_k^\delta \end{bmatrix}. \tag{38}$$

As in the delay free case, assuming $j \geq k : j \notin \mathcal{I}^\mathcal{T}$ and that there does not exist an i such that $i \in \mathcal{I}^\tau$ and $t_k < t_i < t_j$, it holds that

$$\begin{bmatrix} \tilde{\xi}_j \\ e_j^\delta \end{bmatrix} = F_\delta^{j-k} \begin{bmatrix} \tilde{\xi}_k \\ e_k^\delta \end{bmatrix}, \text{ with } F_\delta^{j-k} =: \begin{bmatrix} A_{1|j-k}^\delta & B_{1|j-k}^\delta \\ B_{2|j-k}^\delta & A_{2|j-k}^\delta \end{bmatrix}. \tag{39}$$

To derive a detection algorithm for the case with transmission delays, one needs to find a suitable Lyapunov function candidate, assuming a given controller, that can be used to guarantee satisfaction of (8) in Proposition 2. By the same reasoning as in the delay free case, we compute the candidate, if possible, using a positive definite matrix $\bar{P}$ that satisfies the LMIs

$$\left(A_{1|j}^\delta\right)^\intercal \bar{P} A_{1|j}^\delta - (1+\bar{r})\bar{P} \preceq 0, \; j \in \{1, \dots, m+1\} \tag{40}$$

for a fixed $\bar{r} > 0$. One remaining difference for the case with transmission delays is the horizon for the predicted auxiliary state $\tilde{\xi}_k^p$. While in the delay free case the prediction horizon was given by $k + m - \bar{m}_k + 1$, where $\bar{m}_k$ denotes the number of unsuccessful transmissions since the last successful transmission, the prediction horizon for the case with delay is given by $k + m - \bar{m}_k + 2$. The reason is directly given by the delay $\delta = h$, since now we have to predict one step further since a message successfully sent at time t_k implies a successfully received message only at time t_{k+1}. Using those modifications, it is now possible to state Algorithm 2 and the corresponding theorem for the case with transmission delays.

Theorem 17 *Assume that system* (1) *is controlled with controller* (2) *with a given gain K. Assume furthermore that necessary transmissions are detected according to the PETC mechanism specified by Algorithm 2 and that the network guarantees that at most m successive packets are lost and the transmission delay satisfies $\delta = h$. If there exists a positive definite matrix $\bar{P}$ satisfying* (40) *with $\bar{r} > 0$, and the parameters of the triggering mechanism are chosen as $\nu \in \mathbb{N}, \bar{c} > 0$, then the equilibrium at $\xi = 0$ of the DDS* (4) *is exponentially stable in the large.*

Algorithm 2 Triggering mechanism with $\delta = h$.

1: **if** $t \in (t_k)_{k\in\mathbb{N}}$ **then**
2: **if** $t = t_0$ **then**
3: $i^{\text{ref}} \leftarrow 0$, $\bar{m} \leftarrow 0$
4: $\tilde{\xi}^{\text{ref}} \leftarrow \begin{bmatrix} \hat{x}_0 & \hat{x}_{-1} \end{bmatrix}^\intercal$ with some $\hat{x}_{-1} \in \mathbb{R}^n$
5: $V^{\text{ref}} \leftarrow \left(\tilde{\xi}^{\text{ref}}\right)^\intercal \bar{P} \tilde{\xi}^{\text{ref}}$ with $\bar{P}$ s.t. (40) holds
6: $\tilde{\xi}^p \leftarrow \begin{bmatrix} I & 0 \end{bmatrix} F_\delta^{2+m-\bar{m}} \begin{bmatrix} \tilde{\xi}_k \\ e_k \end{bmatrix}$, $V^p \leftarrow \left(\tilde{\xi}^p\right)^\intercal \bar{P} \tilde{\xi}^p$
7: **if** $(k - i^{\text{ref}} \geq \nu)$ OR $(V^p - V^{\text{ref}} > -\bar{c}\|\tilde{\xi}^{\text{ref}}\|^2)$ **then**
8: send x_k over the network and wait for ACK
9: **if** transmission is successful **then**
10: $i^{\text{ref}} \leftarrow k+1$, $\bar{m} \leftarrow 0$, $V^{\text{ref}} \leftarrow \tilde{\xi}_{k+1}^\intercal \bar{P} \tilde{\xi}_{k+1}$
11: **else**
12: $\bar{m} \leftarrow \bar{m} + 1$
13: **else**
14: no transmission of x_k necessary

PROOF. The setup can be described by the DDS (4) and thus Proposition 2 can be applied, where satisfaction of (7) is guaranteed by the same timeout mechanism, employing $\nu \in \mathbb{N}$, as in the case without delay.

It remains to show that (8) holds with the Lyapunov function candidate $V(\xi) = \xi^\intercal \bar{P} \xi$ and some $c > 0$ for all τ_k with $k \in \mathbb{N}$. This is again shown in an iterative fashion using the fact that $\xi(\tau_k) = \tilde{\xi}(\tau_k)$. One knows by definition that a successful transmission is received at time $t = \tau_k$ and assumes without loss of generality that $\tau_k = t_i$ for some $i \in \mathbb{N}$. If the next successful transmission is received after at most $m+1$ time instants, i.e., $\tau_{k+1} \in \{t_{i+1}, \dots, t_{i+m+1}\}$ one knows from the design of the Lyapunov function candidate according to (40) that (8) holds with $c = \bar{r}\lambda_{\min}(\bar{P})$.

Employing the same arguments as in the delay free case, it is always guaranteed that $\tilde{\xi}_{k+1}^p = \tilde{\xi}_k^p$ if triggering is necessary at time t_k but the packet is lost. Thus, for the case that $\tau_{k+1} \notin \{t_{i+1}, \dots, t_{i+m+1}\}$, one knows that the triggering condition cannot be violated at time t_i, since it would be violated at all successive time instants as well and successfully sent at latest at t_{i+m} and received at t_{i+m+1}. Due to the fact that $\tilde{\xi}_i^p = x_{i+2+m}$ and the fact that triggering is not necessary at t_i, the triggering mechanism guarantees, according to line 10 of Algo-

rithm 2 that (8) holds with $c = \bar{c}$ if $\tau_{k+1} = t_{i+m+2}$. The proof is completed by the same iterative argumentation as in the case without delay. □

5.3 Adaption for NCS

In this subsection, we want to investigate the behavior of the event-based sampling mechanisms in case of packet losses and compare it to a desirable behavior in a networked setting. Motivated by that, a possible modification for Algorithms 1 and 2 is described.

In Section 4, a static triggering mechanism was derived, that compares the norm of the transmitted error with the norm of the current state. If the trigger condition is violated but the packet is lost, it is very likely, although not guaranteed, that a transmission is demanded at the next time instant. This is even clearer for the dynamic approaches in this section. We used a prediction over a horizon that gets shortened if a packet is lost. Thus, if the threshold $\bar{c}_x$, respectively $\bar{c}$, is constant, the trigger condition after a packet was lost is equal to the condition at the previous time step and thus retransmissions are implicitly guaranteed. However, when investigating the design of network protocols, e.g. TCP, the possibility of retransmissions is strictly regulated by suitable mechanisms for congestion control, e.g. [2] for TCP. The main reasoning for the interaction of retransmissions and congestion control is the fact that transmissions should be reduced if packet losses occur due to congestion in the network.

A nice property of the dynamic mechanisms with respect to those considerations, is that such an adaption technique can be easily included. The main technical reason is that Theorems 13 and 17 still hold if the constant $\bar{c}_x$, respectively $\bar{c}$, is replaced by a time-varying function which has values that are uniformly lower bounded by a positive constant, i.e., $\bar{c}_x(k) \geq \underline{c}_x > 0$, respectively $\bar{c}(k) \geq \underline{c} > 0$ for all $k \in \mathbb{N}_0$, since the conditions of Corollary 3, respectively Proposition 2, are still satisfied. One possible way to generate those thresholds, for the example of $\bar{c}_x(k)$, is given by fixing a desired bound $\underline{c}_x > 0$, an initial value $\bar{c}_x(0) > \underline{c}_x$ and updating the threshold according to

$$\bar{c}_x(k+1) = \begin{cases} \bar{c}_x(0) & \text{if } k \in \mathcal{I}^{\mathcal{T}}, \\ \bar{c}_x(k) - \gamma_1(\bar{c}_x(k) - \underline{c}_x) & \text{if } k \in \mathcal{I}^{\mathcal{T}^s} \setminus \mathcal{I}^{\mathcal{T}}, \\ \gamma_2 \bar{c}_x(k) & \text{otherwise,} \end{cases} \tag{41}$$

with $\gamma_1 < 1$ and $\gamma_2 > 1$. The benefits of this adaption will become apparent in the following section's numerical example.

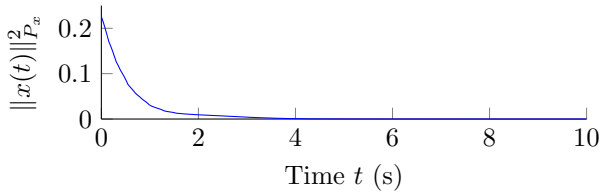

Fig. 2. $\|x\|^2_{P_x}$ with static mechanism ($m = 2$, $\delta = 0$).

6 Simulation results

In this section, an academic example, used in [26] and [12], will be employed to demonstrate the derived mechanisms. Afterwards a comparison concerning the robustness to packet dropouts compared to [21] is given for an inverted pendulum example.

6.1 Demonstration of the derived mechanisms

We consider system (1) with $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \end{bmatrix}^\intercal$, (2) with $K = \begin{bmatrix} 1 & -4 \end{bmatrix}$ and a desired sampling period $h = 0.05$ s. For conducting the simulations, YALMIP (R20150918) [18] and SeDuMi (1.3) [25] were used. We will first investigate the static triggering mechanism derived in Section 4. Thus, the goal is to find a value for σ such that trigger rule (19) guarantees exponential stability in the large, despite at most m successive packet losses. Using Theorem 10, one can show that exponential stability in the large is guaranteed for $\sigma \leq 0.106$ if $m = 1$ and $\sigma \leq 0.036$ if $m = 2$. For the case $m = 3$, it is not possible to find a value for σ such that the conditions of Theorem 10 are satisfied for the given configuration. Fig. 2 shows the evolution of the Lyapunov function for a simulation with $\sigma = 0.036$ and $m = 2$. The simulation initiates at $x_0 = \begin{bmatrix} -0.26 & 0.44 \end{bmatrix}^\intercal$ and we use $\hat{x}_0 = 1.05 x_0$ which demonstrates that no assumption on successful triggering at t_0 is necessary. It can be seen that although the condition was designed based on non-monotonic Lyapunov functions, the resulting evolution of the Lyapunov function decreases strictly monotonically.

The dynamic mechanisms in Section 5 were derived to reduce this conservativity with the goal to trigger less frequently. Indeed, Fig. 3 shows the evolution of the Lyapunov function of a simulation with $m = 5$ where Algorithm 1 is used with a matrix $\bar{P}_x$ satisfying (34) with $\bar{r} = 0.03$ and $\bar{c}_x$ is chosen equal to $\bar{r}\lambda_{\min}(\bar{P}_x)$. It can be seen, that indeed the possibility that the Lyapunov function may increase at some time instants is exploited. For the given setup, matrices $\bar{P}_x$ can be found for up to $m = 10$ consecutive packet losses, while only up to two were possible with the static mechanism.

A third simulation is conducted to demonstrate the

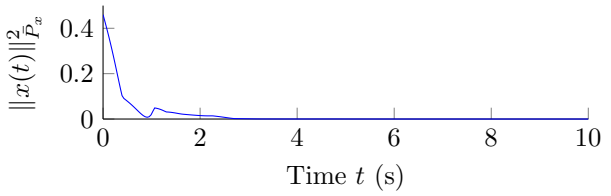

Fig. 3. $\|x\|^2_{\bar{P}_x}$ with dynamic mechanism ($m = 5$, $\delta = 0$).

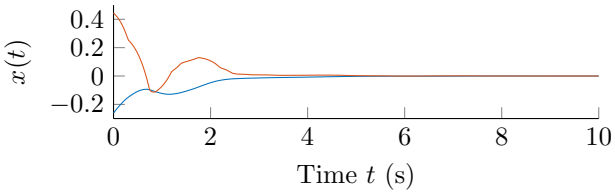

Fig. 4. x with dynamic mechanism ($m = 5$, $\delta = h$).

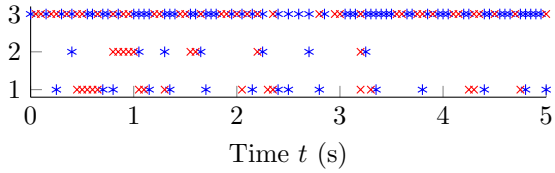

Fig. 5. Triggering times for simulated mechanisms.

mechanism derived in Subsection 5.2, i.e., the case with delay $\delta = h = 0.05$ s, with the adaption technique from Subsection 5.3 being applied. In this case, one can find matrices $\bar{P}$ satisfying (40) with $\bar{r} = 0.03$ according to Theorem 17 for up to $m = 6$ successive packet losses, such that Algorithm 2 together with (41) (in a modified form for the delay case) can be applied. The evolution of the states with $m = 5$, $\bar{c}(0) = 0.5\bar{r}\lambda_{\min}(\bar{P})$, $\gamma_1 = 0.3$, $\gamma_2 = 2$, and $\underline{c} = 10^{-6}$ is shown in Fig. 4.

As a last comparison, the triggering times of all presented simulations are shown in Fig. 5, where blue asterisks denote that a transmission is necessary and the packet is not lost while red crosses denote packet losses. It can be seen that the static mechanism (top line) leads to very frequent transmissions compared to the dynamic ones. Furthermore, effectiveness of the adaption technique derived in Subsection 5.3 can be seen by the fact that without the adaption (middle line), packet losses lead to retransmission attempts while with the adaption (bottom line) this is not necessarily the case. This can help to resolve conflicts in a shared communication medium.

6.2 Comparison of robustness to dropouts with [21]

Consider the inverted pendulum on a cart, modeled as (1) with four states and matrices A and B as in [21] without delay which in the notation of [21] reads as $\bar{\tau} = 0$ and $h = \eta_3 - \bar{\tau} = \eta_3$. We compare the robustness of the triggering strategy according to Table 3 (Table 2 for the controllers) in [21] with the static and dynamic triggering strategy from Sections 4 and 5. Therefore, we denote d_{MANSPL} from [21] as $m_{[21]}$ and the maximal m such that we can apply the static (resp. dynamic) mechanism by m_s (resp. m_d). The results in Table 1 show, at least for all examples based on the controllers designed for the mechanism in [21], the strict relation $m_s < m_{[21]} < m_d$ with $m_s = 0$ for all considered values of h in this particular example. On the one hand this shows that the trigger rule from [21] can tolerate more dropouts than the presented static condition, where one reason could be the fact that the static condition in this article is built upon (19) which offers less degrees of freedom than the static condition in [21]. On the other hand one can see that the dynamic strategy can tolerate the largest number of successive packet loss without a significant increase in the computational effort as explained in Remark 16.

Table 1
Comparison of the maximal number of successive dropouts.

h	K	$m_{[21]}$	m_s	m_d
0.05	[16.22 41.08 516.03 294.91]	2	0	6
0.09	[10.49 28.13 411.72 234.21]	2	0	3
0.13	[12.02 33.64 447.60 254.77]	1	0	2

7 Conclusion

The article showed how non-monotonic Lyapunov functions can be employed to simplify analysis and design of event-triggered strategies for exponentially stabilizing NCS. The essence of the stability results derived in Section 3 is that the main properties that have to be satisfied by the event-triggered system, are uniform upper and lower bounds on the times between successful transmissions and a quadratic decrease of the Lyapunov function in between those times. A static mechanism was derived, but since it is quite conservative this was only done so for the case without delays. It was demonstrated how dynamic triggering mechanisms relying on predictions can be derived directly from the conditions in Proposition 2 and Corollary 3 for systems with and without delay. Since the stability analysis for the dynamic mechanisms does not prohibit an online adjustment of the sensitivity, we were able to introduce an additional adaptive mechanism that shapes the resulting network traffic, in particular concerning the behavior after a packet loss. While the derived mechanisms show a nice behavior in simulations, there are some open points for future research. The influence of possible disturbances needs to be taken into account in the predictions as explained in Remark 14. Furthermore, it would be interesting, whether the results based on non-monotonic Lyapunov functions can be extended to analyze (robust) performance in the same fashion as standard Lyapunov techniques. Since the basic results on non-monotonic Lyapunov functions are not necessarily focusing on linear systems, an extension of the approach to nonlinear systems is a desirable

next step although for example the exact state prediction does not translate to such a setup and a suitable modification has to be derived.

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A General finite dimensional discontinuous dynamical systems

In [20], a finite dimensional DDS is represented by the four-tuple $\{\mathbb{R}_{\geq 0}, \mathbb{R}^n, A, S\}$ consisting of time set $\mathbb{R}_{\geq 0}$, state space $\mathbb{R}^n$, set of initial conditions $A \subseteq \mathbb{R}^n$, and the family of motions S, together with the set of discontinuities E, that may depend on the specific motion $p \in S$. This set is assumed to be unbounded, discrete, and to have no finite accumulation points. In the theorem, the notation $f(r) = \mathcal{O}(r^q)$ as $r \to 0^+$ is used for $\limsup_{r\to 0^+} \frac{|f(r)|}{r^q} < \infty$.

Theorem 18 (Theorem 6.4.7 in [20]) *Let $\{\mathbb{R}_{\geq 0}, \mathbb{R}^n, A, S\}$ be a finite-dimensional DDS and assume that $x_e = 0$ is an equilibrium. Assume that there exist a function $v : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \Rightarrow \mathbb{R}_{\geq 0}$ and four positive constants c_1, c_2, c_3, and b such that*

$$c_1\|x\|^b \leq v(x,t) \leq c_2\|x\|^b \tag{A.1}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$.

Assume that for every $\phi(\cdot, t_0, x_0) \in S$, $v(\phi(t, t_0, x_0), t)$ is continuous everywhere on $\mathbb{R}_{\geq t_0}$ except on an unbounded discrete subset $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\}$ of $\mathbb{R}_{\geq t_0}$ with no finite accumulation points. (E may depend on $\phi(\cdot, t_0, x_0)$.) Furthermore, assume there exists a function $f \in C[\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}]$ with $f(0) = 0$ such that

$$v(\phi(t, t_0, x_0), t) \leq f(v(\phi(\tau_k, t_0, x_0), \tau_k)) \tag{A.2}$$

for all $t \in (\tau_k, \tau_{k+1}), k \in \mathbb{N}_0$, where $\tau_0 := t_0$, and such that for some positive constant q, f satisfies

$$f(r) = \mathcal{O}(r^q) \text{ as } r \to 0^+. \tag{A.3}$$

Assume that for all $k \in \mathbb{N}_0$

$$\frac{1}{\tau_{k+1} - \tau_k}\left[v(\phi(\tau_{k+1}, t_0, x_0), \tau_{k+1}) - v(\phi(\tau_k, t_0, x_0), \tau_k)\right] \leq -c_3\|\phi(\tau_k, t_0, x_0)\|^b \tag{A.4}$$

for all $x_0 \in A$ and $\phi \in S$. Then the equilibrium $x_e = 0$ of the DDS is exponentially stable in the large.

B Proof for Proposition 2

PROOF. The proof is divided into two parts. In the first part, one uses the bound $\tau_1 - \tau_0 \leq \overline{\eta}$ to derive a bound on $\|\phi(t, t_0, \xi_0)\|$ for $t \in [t_0 = \tau_0, \tau_1]$. The second part uses an auxiliary DDS with the same dynamics as in (4) where the discontinuities are at $\mathcal{T} \setminus \{\tau_1\} = \{\tau_2, \tau_3, \dots\}$ and the initial condition is contained in a restricted set. Then we can show that under the given assumption, the zero solution of this auxiliary DDS is exponentially stable in the large and together with the bound on $\|\phi(t, t_0, \xi_0)\|$ for $t \in [t_0, \tau_1]$, we can conclude exponential stability in the large of the equilibrium $\xi_e = 0$ of (4).

Due to the linearity of the continuous dynamics in (4), one can compute the unique solution $\phi(t, t_0, \xi_0)$ for $t \in [t_0, \tau_1)$ and $\xi \in \mathbb{R}^{2n}$ as $\phi(t, t_0, \xi_0) = \mathrm{e}^{A_\xi(t-t_0)}\xi_0$. With $\mu(A_\xi) := \lambda_{\max}\left(\frac{1}{2}(A_\xi^T + A_\xi)\right)$, that has the property $\|\mathrm{e}^{A_\xi s}\| \leq \mathrm{e}^{\mu(A_\xi)s}$ for all $s \geq 0$ as shown in [6], one can derive $\|\phi(t, t_0, \xi_0)\| = \|\mathrm{e}^{A_\xi(t-t_0)}\xi_0\| \leq \mathrm{e}^{(t-t_0)\mu(A_\xi)}\|\xi_0\| \leq \max\{e^{\mu(A_\xi)\overline{\eta}}, 1\}\|\xi_0\|$ for all $t \in [t_0, \tau_1)$. Furthermore $\|\phi(\tau_1, t_0, \xi_0)\| = \|B_\xi\phi(\tau_1^-, t_0, \xi_0)\|$ and thus $\|\phi(\tau_1, t_0, \xi_0)\| \leq \|B_\xi\| \max\{e^{\mu(A_\xi)\overline{\eta}}, 1\}\|\xi_0\|$. One can conclude for all $t \in [t_0, \tau_1]$ that

$$\|\phi(t, t_0, \xi_0)\| \leq \underbrace{\max\{1, \|B_\xi\|\} \max\{e^{\mu(A_\xi)\overline{\eta}}, 1\}}_{k'} \|\xi_0\|. \tag{B.1}$$

The second part of the proof employs an auxiliary DDS with

$$\begin{aligned} \dot{y} &= A_\xi y & \tau_k \leq t \leq \tau_{k+1}, k \in \mathbb{N} \\ y(t) &= B_\xi y(t^-) & t \in \mathcal{T} \setminus \{\tau_1\} \end{aligned} \tag{B.2}$$

with $y_0 := y(\tau_1) = \phi(\tau_1, t_0, \xi_0)$ and thus $y_0 \in A_y := \{\phi(\tau_1, t_0, \xi_0) | \xi_0 \in \mathbb{R}^{2n}\}$ and the set of discontinuities consists of $E_y = \{\tau_2, \tau_3, \dots\}$. Note that with this setup $\phi_y(t, \tau_1, y_0) = \phi(t, t_0, \xi_0)$ for $t \geq \tau_1$, where ϕ_y is the solution of (B.2). One can now apply Theorem 18 with $v(y,t) = V(y) = y^\intercal P y$, where (A.1) holds with $c_1 = \lambda_{\min}(P)$ and $c_2 = \lambda_{\max}(P)$. Using the same derivations as for the case $[\tau_0, \tau_1)$ before, we can show that (A.2) holds for all $t \in (\tau_k, \tau_{k+1}), k \in \mathbb{N}$ with $f(r) = \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \max\left\{\mathrm{e}^{\mu(A_\xi)\bar{\eta}}, 1\right\}^2 r$ satisfying (A.3) for $q = 1$. Furthermore, with (7) and (8) and the definition of A_y, one notices that (A.4) holds for $b = 2$ and $c_3 = \frac{c}{\bar{\eta}}$. Thus we know, using Theorem 18, that the zero solution of the auxiliary DDS is exponentially stable in the large, i.e., according to Definition 1, there exist an $\tilde{\alpha} > 0$ and a $\tilde{\gamma} > 0$, and for any $k'\beta > 0$ (and thus for any $\beta > 0$),

there exists a $\tilde{k} > 0$, dependent on β, such that for $t \geq \tau_1$, $\|y_0\| < k'\beta \Rightarrow \|\phi(t, t_0, \xi_0)\| \leq \tilde{k}\|y_0\|^{\tilde{\gamma}} \mathrm{e}^{-\tilde{\alpha}(t-\tau_1)}$. Note that this implication shows that $\tilde{k}$ and $\tilde{\gamma}$ satisfy the property

$$c \in [0, k'\beta) \Rightarrow c \leq \tilde{k}c^{\tilde{\gamma}} \tag{B.3}$$

for some auxiliary real number c.

The two parts are now used to prove the proposition. From (B.1) one knows that for any $\beta > 0$, $\|\xi_0\| < \beta \Rightarrow \|\phi(\tau_1, t_0, \xi_0)\| = \|y_0\| \leq k'\|\xi_0\| < k'\beta$ and thus for $t \geq \tau_1$

$$\begin{aligned}\|\xi_0\| < \beta \Rightarrow \|\phi(t, t_0, \xi_0)\| &\leq \tilde{k}\|y_0\|^{\tilde{\gamma}} \mathrm{e}^{-\tilde{\alpha}(t-\tau_1)} \\ &\leq \tilde{k}k'^{\tilde{\gamma}} \|\xi_0\|^{\tilde{\gamma}} \mathrm{e}^{-\tilde{\alpha}(t-\tau_1)}. \end{aligned} \tag{B.4}$$

Since it was shown that for any $\beta > 0$ the implication $\|\xi_0\| < \beta \Rightarrow \|\phi(t, t_0, \xi_0)\| < k'\beta$ holds for all $t \in [t_0, \tau_1]$, we can use (B.3) to show that for $t \in [t_0, \tau_1]$, it holds that

$$\begin{aligned}\|\xi_0\| < \beta \Rightarrow \|\phi(t, t_0, \xi_0)\| &\leq \tilde{k}\|\phi(t, t_0, \xi_0)\|^{\tilde{\gamma}} \leq \tilde{k}k'^{\tilde{\gamma}} \|\xi_0\|^{\tilde{\gamma}} \\ &\leq \tilde{k}k'^{\tilde{\gamma}} \mathrm{e}^{\tilde{\alpha}\bar{\eta}} \|\xi_0\|^{\tilde{\gamma}} \mathrm{e}^{-\tilde{\alpha}(t-t_0)}. \end{aligned} \tag{B.5}$$

Thus, the combination of (B.4) and (B.5) shows that there exist $\alpha > 0, \gamma > 0$ and for any $\beta > 0$, there exists $k > 0$ such that $\|\xi_0\| < \beta \Rightarrow \|\phi(t, t_0, \xi_0)\| \leq k\|\xi_0\|^{\gamma} \mathrm{e}^{\alpha(t-t_0)}$ holds for all $t \geq t_0$ with $\gamma = \tilde{\gamma}$, $\alpha = \tilde{\alpha}$, and $k = \tilde{k}k'^{\gamma} \mathrm{e}^{\alpha\bar{\eta}}$. □

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