Implicit solutions to constrained nonlinear output regulation using MPC

Johannes Köhler¹, Matthias A. Müller², Frank Allgöwer¹

Abstract—In this paper, we show that a simple model predictive control (MPC) scheme can solve the constrained nonlinear output regulation problem without explicitly solving the classical regulator (Francis-Byrnes-Isidori) equations. We first study the general problem of stabilizing a set with MPC using a positive semidefinite (input/output) cost function under suitable stabilizability and detectability assumptions, similar to Grimm et al. (2005) [1]. We show that in the output regulation setting, these conditions hold, if the nonlinear constrained regulation problem is (strictly) feasible, the plant is incrementally stabilizable, incrementally input-output to state stable (i-IOSS) and the control input can be uniquely reconstructed from the plant/reference output. Given these structural assumptions, by simply penalizing the predicted output error in the MPC stage cost, the closed loop implicitly stabilizes a state trajectory that solves the regulator equations, if a sufficiently large prediction horizon is used.

I. INTRODUCTION

Motivation: Output regulation is one of the standard problems in control theory, encompassing dynamic output tracking and disturbance attenuation in a common framework [2, Ch. 8]. The classical approach is to solve the regulator (Francis-Byrnes-Isidori) equations [3], which transforms the problem to the stabilization of a given reachable state and input trajectory. While there has been tremendous research on, e.g., necessary and sufficient conditions for the solvability of the regulator equations [3], the numerical solution of the regulator equations still poses a bottleneck for complex nonlinear systems. In this paper, we present a solution to the output regulation problem using model predictive control (MPC) [4], which does not require the solution to the regulator equations.

Related work: Designing an MPC scheme to stabilize a given steady-state can be achieved with or without terminal ingredients [4], [5] and [6], [7]. Stabilizing a given feasible dynamic state and input trajectory can be similarly achieved, compare [8], [9] and [10]. In case the exosystem only generates constant trajectories, the output regulation problem has received considerable attention in the MPC research in the context of setpoint tracking and offset free tracking, compare e.g. [11], [12]. Similarly, in case the external signals are periodic with a known period length T, the output regulation problem can be solved by computing the optimal T-periodic trajectory offline/online [13]/[14]. In [15] output regulation for SISO plants with a well-defined relative degree is studied and a local (polynomial) approximation to the regulator (Francis-Byrnes-Isidori) equations is employed to obtain suitable terminal ingredients, but no closed-loop guarantees are obtained. In [16] and [17], the issue of additional unpredictable disturbances in the exosystem are considered. Essentially, all of these approaches rely on some means to transform the problem to the stabilization of a given state and input trajectory. In this paper, we circumvent this problem using an analysis based on detectability [1].

Contribution: We study an MPC scheme that solves the output regulation problem by simply minimizing the predicted output error $y^\ell$. We provide sufficient conditions, such that the resulting closed loop stabilizes the regulator manifold, i.e., solves the nonlinear regulator problem. In particular, we consider the following structural assumptions: a) the regulator equations admit a strictly feasible solution, b) the plant is incrementally stabilizable, c) the plant is incrementally input-output to state stable (i-IOSS) and d) the input reference can be uniquely reconstructed from the output reference. Under these structural conditions we can ensure that the proposed simple MPC scheme solves the regulator problem by implicitly stabilizing the (unknown) trajectory that solves the regulator equations, if a sufficiently large prediction horizon $N$ is used.

Outline: Section II presents the general theory regarding set stabilization with MPC using a positive semidefinite stage cost $\ell$. Section III presents the considered constrained output regulation problem and shows stability of the regulator manifold. Section IV concludes the paper.

II. MPC - SEMIDEFINITE COSTS AND SET STABILIZATION

In this section, we study stability of nonlinear MPC schemes with positive semidefinite cost functions without any terminal ingredients (no terminal cost or terminal set), similar to [1].

Setup: We consider a nonlinear discrete-time system

$$x_{t+1} = f(x_t, u_t),$$

with the state $x \in \mathbb{X} \subseteq \mathbb{R}^n$, control input $u \in \mathbb{U} \subseteq \mathbb{R}^m$, time step $t \in \mathbb{N}$, and dynamics $f: \mathbb{X} \times \mathbb{U} \to \mathbb{X}$. The system is subject to point-wise in time constraints on the state and input $(x_t, u_t) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$. We consider a positive semidefinite stage cost $\ell: \mathbb{Z} \to \mathbb{R}_{\geq 0}$ and the following MPC
optimalization problem:

\[ V_N(x_t) := \inf_{u_{t-1}, \ldots, u_0 \in U^N} J_N(x_{0:t}, u_{0:t}) := \sum_{k=0}^{N-1} \ell(x_{k|t}, u_{k|t}), \]

\[ \text{s.t. } x_{0|t} = x_t, \]
\[ x_{k+1|t} = f(x_{k|t}, u_{k|t}), \]
\[ (x_{k|t}, u_{k|t}) \in Z, \]
\[ k = 0, \ldots, N - 1. \]

For simplicity we assume \( f, \ell \) is continuous and \( U \) compact. The solution to this optimization problem is the value function \( V_N \) and optimal state and input trajectories \((x_{*|t}, u_{*|t})\). The resulting closed-loop system is given by

\[ u_t = u_{0|t}^*, \quad x_{t+1} = f(x_t, u_{0|t}^*) = x_{*|t+1}, \quad t \geq 0. \]

Similar to [1], we consider a (continuous) positive semidefinite function \( \sigma : X \to \mathbb{R}_{\geq 0} \) as a state measure to be minimized, while satisfying the constraints. To analyse the properties of the closed loop, we consider a stabilizability and detectability condition, similar to [1, SA3/4].

**Assumption 1.** (Local stabilizability) There exist constants \( \gamma_s, \delta_s > 0 \), such that for any \( x \in X_s := \{ x \in X \mid \sigma(x) \leq \delta_s \} \), Problem (2) is feasible and the value function satisfies

\[ V_N(x) \leq \gamma_s \sigma(x), \quad \forall x \in \mathbb{N}. \]

Compared to [1], Assumption 1 only assumes local stabilizability, which is significantly easier to verify and also applicable to unstable systems and/or in the presence of state constraints, which are not control invariant. In case \( \sigma(x) = \|x\|^2 \), Assumption 1 corresponds to the local “controllability” condition in [7, Ass. 1], which is less restrictive than the exponential “controllability” condition used in [6, Ass. 3.5].

Similar local bounds on the value function are used in tracking MPC with and without terminal constraints in [7, Ass. 2] and [10, Prop. 2], respectively.

**Assumption 2.** (Detectability) There exists a function \( W : X \to \mathbb{R}_{\geq 0} \) and constants \( \gamma_o, \epsilon_o > 0 \), such that

\[ W(x) \leq \gamma_o \sigma(x), \]
\[ W(f(x, u)) - W(x) \leq -\epsilon_o \sigma(x) + \ell(x, u), \]

for any \((x, u) \in Z\).

Assumption 2 is a special case of the strict dissipativity condition typically used in economic MPC [18]. The main difference is that \( W \) satisfies the upper bound (5a), while in economic MPC only boundedness (from below) of \( W \) is assumed (c.f. [19]). This small technical difference is the main reason that the analysis of economic MPC schemes and the resulting performance bounds are significantly more conservative, compare [20]. Note that Assumption 2 is trivially satisfied with \( W = 0 \) if \( \ell(x, u) \geq \sigma(x) \), which is the standard case studied in most of the MPC literature [6, 7, 8, 13].

In Section III, we discuss Assumptions 1–2 in detail, providing sufficient conditions based on incremental system properties (c.f. Prop. 2–3).

**Theoretical analysis:** The following theoretical analysis combines ideas from [1, Thm. 1–2] to deal with positive semidefinite stage costs \( \ell \) using a detectability condition (Ass. 2) with the methods in [10, Thm. 1–2] to consider less restrictive local stabilizability conditions (Ass. 1).

**Theorem 1.** Let Assumptions 1–2 hold. For any constant \( \bar{\gamma} > 0 \), there exist constants \( N_{\bar{\gamma}}, \gamma_{\bar{\gamma}} > 0 \), such that for \( N > N_{\bar{\gamma}} \) and initial condition \( x_0 \in X_{\bar{\gamma}} := \{ x \in X \mid V_N(x) + W(x) \leq \bar{\gamma} \} \), the closed loop satisfies the posed constraints and the function \( Y_N := V_N + W \) satisfies

\[ \epsilon_o \sigma(x_t) \leq Y_N(x_t) \leq \gamma_{\bar{\gamma}} \sigma(x_t), \]
\[ Y_N(f(x_t, u_t)) - Y_N(x_t) \leq -\alpha_N \epsilon_o \sigma(x_t), \]

with

\[ \alpha_N := 1 - \frac{\gamma_{\bar{\gamma}}^2}{\epsilon_o^2 (N - 1)} > 0. \]

**Proof.** Abbreviate \( \ell_k = \ell(x_{k|t}, u_{k|t}) \).

**Part I.** The lower bound in (6a) follows with \( \ell_0 \geq 0 \), and

\[ Y_N(x_t) \leq \gamma_{\bar{\gamma}} \sigma(x_t) \leq \epsilon_o \sigma(x_t) + W(x_t) \leq \epsilon_o \sigma(x_t) + W(x_{*|t}) \]

for any \( x_t \in X_{\delta} \), we directly obtain the bound \( Y_N(x_t) \leq (\gamma + \epsilon_o) \sigma(x_t) \) using (4), (5a). The upper bound in (6a) holds with \( \gamma_{\bar{\gamma}} := \max \{ \gamma_{\bar{\gamma}}, \gamma_o, \bar{\gamma} \} \) using this bound, \( x_t \in X_{\bar{\gamma}} \) and a case distinction whether or not \( x_t \in X_{\delta} \), similar to [10, Thm. 2], [7].

**Part II.** The detectability condition (Ass. 2) implies

\[ W(x_{*|t+1}) - W(x_t) = \sum_{k=0}^{N-1} W(x_{k+1|t}) - W(x_{k|t}) \leq \sum_{k=0}^{N-1} \epsilon_o \sigma(x_{k|t}) + \sum_{k=0}^{N-1} \ell_k. \]

Using \( W(x_{*|t+1}) \geq 0 \) and \( x_t \in X_{\bar{\gamma}} \), we arrive at

\[ \epsilon_o \sum_{k=0}^{N-1} \sigma(x_{k|t}) \leq Y_N(x_t) \leq \min \{ \bar{\gamma}, \gamma_{\bar{\gamma}} \sigma(x_t) \}. \]

Thus, there exists a \( k_2 \in \{ 1, \ldots, N - 1 \} \), such that

\[ \sigma(x_{k_2|t}) \leq \frac{\min \{ \bar{\gamma}, \gamma_{\bar{\gamma}} \sigma(x_t) \}}{\epsilon_o (N - 1)}. \]

Given \( N \geq N_0 := 1 + \frac{\bar{\gamma}}{\epsilon_o \gamma_o} \), this implies \( x_{k_2|t} \in X_{\delta} \). Thus, Assumption 1 ensures that starting at \( x_{k_2|t} \) there exists a feasible state and input trajectory satisfying the bound (4), which implies

\[ V_N(x_{k_2}) + \ell_k \leq \sum_{k=0}^{k_2-1} \ell_k + V_N(x_{k_2}) \]

\[ \leq \sum_{k=0}^{k_2-1} (\epsilon_o (N - 1) \sigma(x_t)). \]

Combining (10) and (5b), the function \( Y_N \) satisfies (6b).

Furthermore, \( \gamma_{N, \gamma_{\bar{\gamma}}} > 0 \) follows from (7), using \( N > N_1 := 1 + \gamma_{\bar{\gamma}} \gamma_{\gamma_{\bar{\gamma}}} / \epsilon_o \). All the arguments hold with \( N > N_{\bar{\gamma}} := \max \{ N_0, N_1 \} \). In addition, inequality (6b) ensures \( Y_N \) is nonincreasing and thus \( x_t \in X_{\bar{\gamma}} \) for all \( t \geq 0 \). 

\[ \square \]
Discussion: Note that Inequalities (6) directly imply 
\( \lim_{t \to \infty} \sigma(x_t) = 0 \). Furthermore, in case \( \sigma(x) = \|x\|_a \), \( a > 0 \), Theorem 1 ensures exponential stability of \( x = 0 \).

In Section III, as on of the main results of the paper, we show how the results in Theorem 1 can be used to design an MPC scheme that implicitly solves the constrained output regulation problem. In the following, we briefly highlight the relevance of Thm. 1 to the problem of MPC for input-output stage costs and models.

**Remark 1.** Similar to the considered problem of output regulation, input-output stage costs \( \ell = \|y\|^2 + \|u\|^2 \) appear naturally in the case of trajectory tracking or path following, if only some output reference is specified as opposed to a full state reference [9]. Likewise, in case an input-output model resulting from some system identification is used, e.g. input-output LPV systems [21], the consideration of positive semidefinite input-output stage costs is natural. In all these results, a given state trajectory is considered and a suitable terminal cost (and terminal region) is constructed offline. Using Theorem 1, the need for constructing such terminal ingredients or determining the corresponding state trajectory can be dropped by choosing a sufficiently large \( N \).

The need to use a positive semidefinite input-output stage cost \( \ell \) and avoid terminal ingredients or state references also appears in data-driven MPC, where (linear) models are implicitly represented using data [22], [23]. However, offline verification of the corresponding detectability and stabilizability conditions (Ass. 1, 2) with such an implicit data-based model is still open research topic, compare e.g. [24] where some input-output system properties are verified for such data based models.

### III. Constrained Output Regulation Amidst Classical Results and MPC

In Section III-A, we present the output regulation problem, including the classical results. In Section III-B, we present the proposed MPC scheme and show that it corresponds to the setup in Section II. In Section III-C, we derive sufficient increment system properties for the rather abstract stabilizability and detectability conditions (Ass. 1–2), culminating in the main theoretical result in Corollary 1. The special case of linear systems is considered in Section III-D.

#### A. Output regulation - classical results

We consider the following nonlinear discrete-time system

\[
\begin{align*}
  x_{t+1}^p &= f^p(x_t^p, u_t, w_t), \\
  w_{t+1} &= s(w_t), \\
  y_t^e &= h^p(x_t^p, u_t) - r(w_t),
\end{align*}
\]

with \( f^p \), \( s \), \( h^p \), \( r \) continuous. The plant dynamics are described by equation (11a) with the plant state \( x^p \in \mathbb{R}^{n_p} \) and the control input \( u \in \mathbb{R}^m \). The exogenous signal \( w \in \mathbb{R}^q \) is generated by the exosystem (11b) and represents both disturbances affecting the plant (11a) and desired reference values (11c). Equation (11c) describes a reference tracking error \( y_e^c \in \mathbb{R}^p \), which is the difference between the plant output \( h^p(x^p, u) \) and the output reference \( r(w) \). The exosystem state is assumed to be contained in some positive invariant set \( \mathbb{W} \), i.e., \( s : \mathbb{W} \to \mathbb{W} \). The control goal is to achieve output nulling \( (\lim_{t \to \infty} \|y_e^c\| = 0) \), while the plant state and control input are supposed to satisfy general nonlinear constraints of the form \( (x_t^p, u_t) \in \mathbb{Z}^p \subseteq \mathbb{X}^p \times U \subseteq \mathbb{R}^{n_p+m} \). The output regulation problem is a special case of the problem considered in Section II with \( x := (x^p, w) \in \mathbb{X} := \mathbb{X}^p \times \mathbb{W} \subseteq \mathbb{R}^{n_p+q} \), \( f(x, u) := (f^p(x^p, u), s(w)) \), \( \mathbb{Z} := \{(x^p, w, u) \in \mathbb{X} \times U | (x^p, u) \in \mathbb{Z}^p \} \).

The classical solution to the output regulation problem is to find functions \( \pi_x : \mathbb{W} \to \mathbb{X}^p \), \( \pi_u : \mathbb{W} \to U \), which satisfy

\[
\begin{align}
  \pi_x(s(w)) &= f^p(\pi_x(w), \pi_u(w), w), \\
  0 &= h^p(\pi_x(w), \pi_u(w)) - r(w),
\end{align}
\]

for any \( w \in \mathbb{W} \). Equations (12) are called the discrete-time regulator equations. In [3] it was shown that the regulator equations are solvable, if the zero dynamics of the plant is hyperbolic. Given a solution to the regulator equations (12), output regulation can be reduced to the problem of stabilizing the error \( e_p^c := x^p - \pi_x(w) \) and a corresponding measure is given by \( \sigma(x) := ||x^p - \pi_x(w)||^2 = ||e_p^c||^2 \).

**Assumption 3.** (Regulator equations) The regulator equations (12) admit a solution \( \pi_x, \pi_u \) and \( (\pi_x(w), \pi_u(w)) \in \text{int}(Z^p) \) for all \( w \in \mathbb{W} \).

### IV. Local Incremental Exponential Stabilizability

There exist constants \( c_{s,t}, c_{u}, \delta_{loc}>0 \) and \( \rho_s \in (0,1) \), such that for any trajectory satisfying \( (x_t^p, v_t, w_t) \in Z^p \times \mathbb{W} \) and \( x_{t+1}^p = f^p(x_t^p, v_t, w_t) \), (11b) for all \( t \geq 0 \), there exists a continuous incremental Lyapunov function \( V_s : \mathbb{X}^p \times \mathbb{N} \to \mathbb{R}_{\geq 0} \) and a Lipschitz continuous feedback \( k : \mathbb{X}^p \times \mathbb{N} \to U \) with \( V_s(x_t^p, t) = 0 \), \( k(x_t^p, t) = v_t \) satisfying the following inequalities for all \( x^p_t, t \) ≤ \( \delta_{loc} \), and all \( t \geq 0 \)

\[
V_s(f^p(x^p, k(x^p, t)), w_t, t + 1) \leq \rho_s V_s(x^p, t), \\
c_{s,t} ||x^p - z^p||^2 \leq V_s(x_t^p, t) \leq c_{s,u} ||x^p - z^p||^2.
\]

### Proposition 1. Let Assumptions 3–4 hold. Consider \( V_s, \kappa \) corresponding to the trajectory \( (x_t^p, u_t) = (\pi_x(w_t), \pi_u(w_t)) \).

There exists a constant \( \delta > 0 \), such that for any initial condition \( x_0^p \) satisfying \( V_s(x_0^p, 0) \leq \delta \), the feedback \( u_t = \kappa(x_t^p, t) \) ensures (uniform) exponential stability of the origin \( e^p = 0 \) and satisfies the constraints, i.e., \( (x_t^p, u_t) \in \mathbb{Z}^p \) for all \( t \geq 0 \).

**Proof.** First, note that \( e_t^p = x_t^p - \pi_x(w_t) = x_t^p - z_t^p \). Due to Ass. 3, we know that \( (z_t^p, v_t) = (\pi_x(w_t), \pi_u(w_t)) \in \text{int}(Z^p) \). For \( \delta \leq \delta_{loc} \), applying Inequality (13a) repeatedly ensures \( c_{s,t} ||e_t^p||^2 \leq V_s(x_t^p, t) \leq \rho_s V_s(x_0^p, 0) \leq c_{s,u} ||e_0^p||^2 \) and thus exponential stability of \( e^p = 0 \). Given that \( (z_t^p, v_t) \in \text{int}(Z^p) \).

\(^1\)Although equation (11c) corresponds to the classical output regulation problem [3], the results presented in this paper can be directly extended to the case where \( y_e^c = h(x^p, u, w) \).
(Ass. 3), there exists a constant $\epsilon_s > 0$, such that $(x^p_t, u_t) \in Z^p$ if $\| (x^p_t - z^p_t, u_t - v_t) \| \leq \epsilon_s$. Due to continuity of $\kappa$ and $V_s$, there exists a constant $\delta \in (0, \delta_{loc})$, such that $V_s(x^p_t, t) \leq \delta$ implies $(x^p_t, \kappa(x^p_t, t)) \in Z^p$, which finishes the proof. \hfill $\square$

Assumption 4 ensures that there exists some feedback to exponentially stabilize any feasible trajectory $(z^p, v^p)$, which implies that it can guarantee stabilizability of the trajectory $(\pi_w(w), \pi_u(w))$. Similar incremental stabilizability conditions have been considered in [10] for trajectory tracking MPC. Satisfaction of Assumption 4 can either be directly verified numerically or by designing a suitable incremental Lyapunov function $V_s$ offline, e.g. using control contraction metrics [25], quasi-LPV design [8], [26], backstepping [27], or feedback linearization. We point out that Proposition 1 only provides a local solution to the constrained output regulation problem and requires knowledge of $\pi_w, \pi_u$, the solution to the regulator equations (12). Both of these restrictions will be relaxed in the proposed MPC in Section III-B.

### B. Proposed MPC scheme

In the following, we present the proposed MPC scheme. The proposed MPC scheme is based on the optimization problem (2) with the following stage cost

$$\ell(x, u) := \| h^p(x^p, u) - r(w) \|^2 = \| y^f \|^2,$$

which penalizes the reference tracking error.

In order to solve the optimization problem (2) with the stage cost (14), we need to be able to predict both the plant state $x^p$ and the exosystem state $w$, and thus assume that $x = (x^p, w)$ can be measured\(^2\) and an accurate prediction model is available. However, we do not need to solve the regulator equations (12). Note that this is only possible, since we do not use a positive definite stage cost $\ell$ or terminal ingredients, both of which would drastically simplify the theoretical analysis but would necessitate knowledge of $\pi_w(u)$. Thus, compared to Proposition 1, the proposed MPC scheme has the following advantages:

- No explicit solution to the regulator equations (12) is required.
- No explicit stabilizing controller $\kappa$ (Ass. 4) is needed.
- MPC scheme enjoys a larger region of attraction.

Since the MPC problem (2) is already analysed in Theorem 1, we only need to show satisfaction of Assumptions 1–2 to ensure that the proposed MPC scheme solves the constrained regulator problem, which will be proven in Corollary 1 in Section III-C.

We emphasize that we do not pose any periodicity conditions on $w$, as e.g. done in [13], [2] or restrict ourselves to constant values $w$ as done in [11], [12].

### C. Incremental system properties

In the following, we derive sufficient conditions for Assumptions 1–2 for the regulator problem (11) and then prove exponential stability of the regulator manifold in Corollary 1.

\(^2\)Classically, the output regulation problem considers error-feedback [2, Ch. 8]. Extending the proposed MPC scheme to the error-feedback setup using (robust) output-feedback MPC designs [28] is an open issue.

### Stability (Ass. 1):

**Proposition 2.** Let Assumptions 3–4 hold and suppose that $h^p$ is locally Lipschitz continuous. Then Assumption 1 holds.

**Proof.** In Proposition 1 it was already shown that $u_t = \kappa(x^p_t, t)$ is a feasible control input for $V_s(x^p_0, 0) \leq \delta$. Thus the optimization problem (2) is feasible with this candidate input for all $x_t \in \mathcal{X}_d$ with $\delta_s := \delta / c_{\epsilon_s}$. It remains to show that the bound (4) holds. Similar to [10, Prop. 2], inequalities (13) and $\kappa$ being Lipschitz imply that there exists a constant $c > 0$ such that

$$\| (x^p_{k^i+1}, u_{k^i}) - (\pi_x(w_{k^i}), \pi_u(w_{k^i})) \|^2 \leq c \rho_s^k \| x^p_t - \pi_x(w_t) \|^2.$$  

Lipschitz continuity of $h^p$ with some Lipschitz constant $L_h$ and $h^p(\pi_x(w_{k^i}), \pi_u(w_{k^i})) \equiv r(w_{k^i})$ imply

$$V_N(x_k) \leq \sum_{k=0}^{N-1} \| h^p(x^p_{k^i+1}, u_{k^i}) - r(w_{k^i}) \|^2$$

$$\leq c L_h^2 \| x^p_t - \pi_x(w_t) \|^2 \sum_{k=0}^{N-1} \rho_s^k \leq \frac{c L_h^2}{1 - \rho_s} \| x^p_0 - \pi_x(w_0) \|^2.$$ \hfill $\square$

Note that Ass. 4 could be relaxed to only hold for $(x^p_t, u_t) = (\pi_x(w_t), \pi_u(w_t))$, which is less restrictive. However, the benefit of Ass. 4 is that it can be verified without solving the regulator equations (12), which is one of the main motivations of the presented work.

### Detectability (Ass. 2): Intuitively, satisfying the detectability condition (Ass. 2) with the stage cost $\ell$ (14) without additional restrictions on the functions $\pi_x, \pi_u$ in (12) requires that we can asymptotically reconstruct the state trajectory $\pi_x(w)$ from the output $y^f$. One standard property used to characterize the detectability of nonlinear systems is incremental input-output-to-state stability (i-IOSS) [29], [30].

**Assumption 5.** (exponential i-IOSS) There exist constants $c_{o,1}, c_{o,u}, c_{o,1}, c_{o,2} > 0$, $\rho_o \in (0, 1)$, such that for any trajectory satisfying $(z^p_t, v_t, w_t) \in Z^p \times \mathbb{W}$ and $z^p_{t+1} = f^p(z^p_t, v_t, w_t)$, (11b) for all $t \geq 0$, there exists a continuous i-IOSS Lyapunov function $V_o : \mathbb{X}^p \times \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$c_{o,1}\| x^p_t - z^p_t \|^2 \leq V_o(x^p, t) \leq c_{o,2}\| x^p_t - z^p_t \|^2,$$

$$V_o(f^p(x^p, u, w, t+1) - \rho_o V_o(x^p, t) \leq c_{o,1}\| u - v_t \|^2 + c_{o,2}\| h^p(x^p, u) - h^p(z^p_t, v_t) \|^2.$$ (16b)

for all $(x^p_t, u) \in Z^p$, $t \geq 0$.

Note that while i-IOSS is a standard condition to design full-order state observers [31, Prop. 23], such an observer typically has access to input and output measurements, while in the considered problem of output regulation, we only have access to the desired output $r(w)$, but not to the desired input $u = \pi_u(w)$. A similar problem is studied under the topic of unknown input observers [32], which typically requires that there exists a (causal) left inverse to the input-output dynamics, which necessitates that the output contains a direct feed through and the input can be directly reconstructed from the output.
Assumption 6. (Nonsingular input cost) There exists a constant $c_{o,3} > 0$, such that
\[ \| h^p(x^p, u) - h^p(z^p, v) \|^2 \geq c_{o,3} \| u - v \|^2 \]  
(17)
for any $(x^p, u) \in \mathcal{Z}^p$, $(z^p, v) \in \mathcal{Z}^p$.

In essence, Assumption 6 implies that we have access to an input/output reference trajectory (or equivalently, that the input reference is uniquely3 determined by the output reference). The need for Ass. 6, including the terminology nonsingular input cost, will become clearer for the special case of linear systems in Sec. III-D.

Proposition 3. Let Assumptions 3, 5 and 6 hold. Then Assumption 2 holds with $\epsilon$ from (14).

Proof. Assumption 3 implies $(z^p_t, v_t) = (\pi_x(w_t), \pi_u(w_t)) \in \text{int}(\mathcal{Z}^p)$ and thus Assumption 5 ensures that there exists an $i$-IOSS Lyapunov function $V_o(x^p, t)$ satisfying (16). Since the trajectory $(z^p_t, v_t)$ is uniquely determined by $w_0$, we replace the time-dependence by a dependence on $w$, yielding $V_o(x^p, w)$. Consider $W(x) = eV_o(x^p, w)$ with $c = \frac{c_{o,3}}{c_{o,1} + c_{o,2} + c_{o,3}} > 0$. The upper bound (5a) follows directly from (16a) with $c_o = c \cdot c_{o,1}$ and $\sigma(x) = \| x^p - z^p_t \|^2$. Inequality (5b) holds with $\epsilon_0 = (1 - \rho_o) \cdot c \cdot c_{o,1} > 0$ using
\[ W(f(x, u)) - W(x) \]
\[ \leq -(1 - \rho_o)eV_o(x^p, w) + c \cdot c_{o,1} \| u - \pi_u(w_t) \|^2 + c \cdot c_{o,2} \| h^p(x^p, u) - r(w_t) \|^2 \]
\[ \leq -\epsilon_0 \sigma(x) + \ell(x, u). \]

Remark 2. One may be tempted to consider $\sigma(x) = \min_{u \in \Pi} \ell(x, u)$, which satisfies Assumption 2 with $W = 0$, $\epsilon_0 = 0$. However, in this case Inequality (4) in Assumption 1 is typically only satisfied if the regulator reference state $\pi_x(w)$ can be uniquely reconstructed from the output $r(w)$, which is quite restrictive. This special case of tracking a given state-input reference trajectory is treated in [10, Thm. 2].

Combined theorem: Given Prop. 2–3, we can recast Thm. 1 for the output regulation problem (Sec. III-B), using intuitive assumptions on the inherent system properties of the plant.

Corollary 1. Consider the MPC scheme (2) with the output regulation setup (11), (14). Let Assumptions 3–6 hold. Suppose further that $\pi_x$ from Ass. 3 and $h^p$ are Lipschitz continuous. For any constant $Y > 0$, there exists a constant $N_\pi$, such that for $N > N_\pi$ and initial condition $x_0 \in \mathcal{X}_Y := \{ x \in \mathcal{X}| V_N(x) + W(x) \leq Y \}$, the constraints are satisfied and the regulator manifold $A := \{ x | x_p = \pi_x(w) \}$ is exponentially stable for the resulting closed loop.

Proof. First, note that Propositions 2–3 ensure that Assumptions 1–2 hold. Thus, for $N > N_\pi$, with $N_\pi$ from Thm. 1, the bounds (6) hold with $\alpha_N > 0$. Define the set distance $\| x \|_A := \inf_{x \in A} \| x - s \|$. In the following, we show that there exists a constant $c_{\pi} > 0$, such that $c_{\pi} \sigma(x) \leq \| x \|_A^2 \leq \sigma(x)$, which in combination with (6) ensures exponential stability of $A$ using standard Lyapunov arguments. For given $(x^p, w)$, denote some minimizer by $\tilde{w} := \arg \min_{w \in \mathcal{W}} \| \pi_x(\tilde{w}) - (x^p, w) \|$. Given that $\pi_x$ is Lipschitz continuous with Lipschitz constant $L_n$, we have
\[ \sigma(x) = \| x^p - \pi_x(x) \|^2 \]
\[ \leq 2(\| x^p - \pi_x(\tilde{w}) \|^2 + \| \pi_x(\tilde{w}) - \pi_x(x) \|^2) \]
\[ \leq 2 \max\{L_n^2, 1\}(\| x^p, w \| - (\pi_x(\tilde{w}), \tilde{w}))^2 := 1/c_{\pi} \| x \|_A^2, \]
where the first inequality uses $(a + b)^2 \leq 2(a^2 + b^2)$ for any $a, b \in \mathbb{R}$. Furthermore,
\[ \| x \|_A = \| \pi_x(\tilde{w}) - (x - (x^p, w)) \| \leq \| x^p - \pi_x(x) \| = \sqrt{\sigma(x)}, \]
which finishes the proof.

Overall, this result implies that the proposed MPC scheme solves the nonlinear constrained regulation problem if:
(a) The regulator problem is (strictly) feasible (Ass. 3),
(b) The plant is incrementally stabilizable (Ass. 4) and detectable (i-IOSS, Ass. 5),
(c) The plant output contains a feed through term that allows to uniquely reconstruct the plant input (Ass. 6).

The intuition for conditions (a)–(b) for the problem at hand is quite clear, while condition (c) seems somewhat restrictive and may possibly be removed using better arguments. However, we wish to emphasize that the crucial advantage of this result compared to, e.g., classical trajectory stabilization (Prop. 1) is the fact that the solution to the regulator equations $\pi_x, \pi_u$ is not needed for the implementation and instead the closed loop will “find” the regulator manifold. In addition, compared to Prop. 1, the proposed MPC scheme yields a larger region of attraction (despite the presence of hard constraints).

D. Linear systems

Consider the special case of linear systems:
\[ f^p(x^p, u, w) = Ax^p + Bu + Pw, s(w) = Sw, \]
\[ h^p(x^p, u) = Cx^p + Du, r(w) = Qw. \]
The regulator equations (12) reduce to
\[ \Pi S = AP + B\Gamma + P, 0 = C\Pi + DTQ, \]
(18)
with $\pi_x(w) = \Pi w, \pi_u(w) = \Gamma w$. Solvability of (18) can be ensured with the so-called nonresonance condition and the matrices $\Pi, \Gamma$ are unique if $m = p$, compare [33, Lemma 4.1]. Assumption 4 reduces to stabilizability of $(A, B)$ and Assumption 5 to detectability of $(A, C)$ (c.f. [29]). Thus, if the input can be uniquely reconstructed from the output (Ass. 6), i.e., $[C, D] - [C, D] \geq \epsilon \cdot \text{diag}(0_n, I_n)$, Assumption 2 reduces to detectability of

3In particular, given $v(w)$ and Assumptions 3, 6, we can uniquely determine $v = \pi_u(w)$ using $(x^p, v) = \arg \min_{u \in \Pi} \| h^p(x^p, u) - r(w) \|^2$, where $z^p$ is some (in general not unique) minimizer. Thus, Assumption 6 is more restrictive than requiring a direct feed through term in the output, compare also the discussion in Section III-D.

4We note that in [10, Thm. 4], also the case of unreachable trajectories is treated, i.e., when Assumption 3 does not hold.
(A, C) (c.f. Prop. 3). Note that if $Z^p$ is polytopic, the resulting MPC problem (2), is a standard quadratic program.

In the following, we briefly revisit the existing literature for linear systems, to discuss the need for Assumption 6. Consider the simplified linear problem without exosystem ($w = 0$), and a stage cost of the form $f(x, u) = \|x\|^2_{R_x} + \|u\|^2_{R_u}$, with $R_u \geq 0$, $R_x = C^T C \geq 0$. Assumption 6 reduces to $R_x \succ 0$ and Assumption 5 to $(A, C)$ detectable. In [19], the connection between detectability (Ass. 2) and strict dissipativity is investigated. For the linear case, if $R_u \succ 0$ and $(A, C)$ detectable, [19, Corollary 2] ensures satisfaction of Assumption 2, which coincides with the results in Prop. 3. Furthermore, in case $R_u \neq 0$ (Assumption 6 is not satisfied), even if $(A, C)$ is observable/detectable, Assumption 2 is in general not satisfied, compare [19, Example 1]. This shows that relaxing Assumption 6 may be non trivial. In case $X$ compact and $R_u \succ 0$, strict dissipativity can even hold if $(A, C)$ is not detectable [34, Thm. 6.1], but the resulting storage function is in general indefinite and thus Assumption 2 does not hold. The case $R_u \neq 0$ can be treated using the results in [35, Thm. 3], in case the input constraints are active at the optimal steady-state (origin), if an additional complementary slackness condition holds. To summarize, there only exist few results in the literature with $R_u \neq 0$ and Assumption 6 may be necessary to show stability, if we wish to use arguments similar to [1]/Theorem 1 based on the detectability condition in Assumption 2.

IV. CONCLUSION
We have shown that a simple MPC scheme solves the nonlinear constrained regulator problem, given certain structural assumptions on the system and a sufficiently long prediction horizon. Future research focuses on obtaining similar stability guarantees while relaxing the nonsingular input cost assumption.

REFERENCES