Approximate dissipativity and performance bounds for interconnected systems

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***Abstract*— We consider the interconnection of dissipative systems through coupling costs in a distributed economic MPC context. Our goal is to provide a structured dissipativity property for the overall system emerging from the subsystems' local dissiaptivity properties and their interconnection. However, following this bottom-up approach, only in very few cases we can expect to establish dissipativity of the overall system based on a minimal set of assumptions. Hence, in this work we introduce the concept of approximate dissipativity, which still allows us to make statements on the system performance, albeit somewhat unsharp. We verify approximate dissipativity for the overall system under cost interconnection of the subsystems, and we demonstrate how this concept can constructively be exploited when adding a new subsystem to the network.**

I. Introduction

Advances in communication and computation technologies have led to the emergence of increasingly complex and large-scale interconnected systems. Such interconnected systems as, e.g., smart electric grids, groups of autonomous vehicles, or highly automated self-organized factories in the context of Industry 4.0 offer an entire new dimension of opportunities for coordinated control, e.g., optimizing the overall system performance or achieving a certain cooperative objective. On the other hand, control of such large-scale systems is an extremely challenging task, since system dynamics are often nonlinear and subject to state and input constraints, there are couplings of various kinds between the subsystems and, moreover, in many applications the subsystems should be operated in such a way that the real, economic cost is optimized for the overall system.

Distributed model predictive control is an appealing control technique for such complex control tasks, since it naturally answers all of the above challenges. There exists a vast amount of literature on distributed model predictive control for all different kinds of system setups (see [1]–[3] for an overview), however, the economic version thereof has not yet been well studied. The distinguishing feature of economic MPC is that the cost criterion to be optimized can be chosen arbitrarily, and hence may directly represent real economic costs such as profit or operating costs. In the centralized setting, various results on the stability and performance of the economic MPC closed loop are available [3]–[7]. For most of these results, a certain dissipativity property of the system and its cost function plays a crucial role. For distributed economic MPC, only few results are available in the literature. In the recent work [8], convergence to an initially unknown overall optimal steady state is enforced through average constraints. The approaches of [9] and [10] rely on a dissipativity property formulated for the overall system and an iterative distributed MPC scheme requiring infinitely many iterations in each time step. Thereby, the distributed problem can again be analyzed in a centralized fashion and the structured nature of the problem is neglected.

Motivated by the sheer large scale of real-world interconnected systems as well as by the fact that these networks are constantly changing due to subsystems entering and leaving, in this work we follow a bottom-up approach focusing on the local subsystems, their local properties and the interconnection structure. More specifically, we investigate how local dissipativity properties of the individual subsystems carry over to the overall system when getting interconnected through coupling costs. The resulting dissipativity property for the overall system is structured and can therefore be exploited in the design and analysis of *non-iterative* distributed economic MPC algorithms. This setup was already considered in our previous work [11]. There, however, dissipativity of the overall system could only be provided in very special situations. In the work at hand, we introduce the concept of approximate dissipativity, which now allows us to make approximate dissipativity statements for the overall system, while still only relying on a minimal set of assumptions on the local subsystems. Moreover, this new concept can not only be used for analysis, but allows for a neat practical approach to assess the variation of overall approximate dissipativity when adding new subsystems to the network, based on local information only.

Finally we note that the interconnection of dissipative systems has already been investigated in the seminal work [12] and more recently, e.g., in [13]. However, in constrast to these works, in the context of (distributed) economic MPC the interconnection of subsystems is *not* by means of input-output interconnections. Instead, dissipativity enters rather indirectly when analyzing the optimal solution to the MPC optimization problem, c.f. [6], thus requiring different analysis techniques.

The remainder of this work is structured as follows. In Section II, the system setup is detailed. The concept of approximate dissipativity is discussed in Section III, and Section IV verifies this property for the overall system. In Section V we constructively utilize the concept of approximate dissipativity in the scenario of adding a new agent to the network, and finally we discuss our results in Section VI.

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Notation. The interconnection structure between subsystems is described by a weighted directed graph $G = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ with vertices $\mathcal{V} = \{1, \ldots, P\}$ representing the subsystems, edges $\mathcal{E} = \{(i,j) \in \mathcal{V} \times \mathcal{V}\}$ representing the directed interconnection of systems, and the set of edge weights $\mathcal{W} := \{q_{ij} \in \mathbb{R}, q_{ij} \geq 0 \, \forall \, (i,j) \in \mathcal{E}\}$. Denote the set of all neighbors of subsystem i by $\mathcal{N}_i := \{j \in \mathcal{V} \,|\, (i,j) \in \mathcal{E}\}$. A directed graph G is said to be weakly connected if there is a path between every pair of vertices when neglecting the edge orientation. Denote by $B \in \{-1, 0, 1\}^{|\mathcal{V}| \times |\mathcal{E}|}$ the incidence matrix of a directed graph G with elements B_{ve} equalling 1 if vertex v is the head of edge e, -1 if v is the tail of e, and 0 otherwise. For a thorough presentation of algebraic graph theory we refer the reader to [14]. A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite if $\rho(0) = 0$ and $\rho(x) \geq 0$ for all $x \in \mathbb{R}^n$. The function $\rho : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $\rho(0) = 0$ and $\rho(x) > 0$ for all $x \neq 0$. A matrix $A = A^\top \in \mathbb{R}^{n \times n}$ is positive (semi)definite if the quadratic form $x^\top A x$ is positive (semi)definite, and we write $A \succ 0$ ($A \succeq 0$). The operator $\otimes$ denotes the standard Kronecker product, $\mathrm{diag}(\cdot)$ denotes the block diagonal matrix built from the respective arguments, $I_n \in \mathbb{R}^{n \times n}$ denotes the $n \times n$-identity matrix, and $\mathbf{1}_n \in \mathbb{R}^n$ denotes the all ones vector of dimension n.

II. System description and problem statement

In this work, we consider a group of P dynamically decoupled discrete-time nonlinear systems

$$x_i(t+1) = f_i(x_i(t), u_i(t)), \quad x_i(0) = x_{i0}, \tag{1}$$

where $x_i(t) \in \mathbb{X}_i \subseteq \mathbb{R}^n$, $u_i(t) \in \mathbb{U}_i \subseteq \mathbb{R}^m$, $i \in \mathbb{I}_{[1,P]} := \{1, \ldots, P\}$, and $f_i : \mathbb{X}_i \times \mathbb{U}_i \to \mathbb{X}_i$ continuous. Denote the set of admissible state and input pairs by $\mathbb{Z}_i \subseteq \mathbb{X}_i \times \mathbb{U}_i$, and the set of admissible steady states for each subsystem i by $\mathbb{Z}_i^s := \{(x_i, u_i) \in \mathbb{Z}_i \,|\, x_i = f_i(x_i, u_i)\}$.

Coupling between the subsystems is introduced through their economic stage cost functions being separable in a purely local economic cost and coupling costs. Coupling costs are induced by interaction with neighboring subsystems subject to the fixed weighted directed interconnection graph $G = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$. In particular, we consider separable cost functions of the form

$$\ell_i(x_i, u_i, x_{-i}) = \ell_{ii}(x_i, u_i) + \sum_{j \in \mathcal{N}_i} \ell_{ij}(x_i, x_j), \tag{2}$$

with the continuous local economic cost function $\ell_{ii} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, and the continuous coupling cost $\ell_{ij} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ induced by the coupling between subsystem i and its neighbors.By x_{-i} we denote the collection of states of all subsystems that are neighbors of subsystem i. Such coupling costs are found, e.g., in classical multi agent settings representing a common objective, or in the case of subsystems sharing a common resource. Studying the overall system, we consider stacked local subsystems' variables as, e.g., for the state and input vectors $x = [x_1^\top, \ldots, x_P^\top]^\top \in \mathbb{R}^{nP}$, $u = [u_1^\top, \ldots, u_P^\top]^\top \in \mathbb{R}^{mP}$. The corresponding overall system dynamics result in

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0, \tag{3}$$

with admissible state and input pairs $(x(t), u(t)) \in \mathbb{Z} := \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_P$, and $f(x,u) = [f_1(x_1, u_1)^\top, \ldots, f_P(x_P, u_P)^\top]^\top$. Accordingly, the set of admissible steady states of the overall systems is $\mathbb{Z}^s = \mathbb{Z}_i^s \times \cdots \times \mathbb{Z}_P^s$. The overall system becomes interconnected when considering its stage cost given as

$$\ell(x,u) = \sum_{i=1}^{P} \ell_i(x_i, u_i, x_{-i}). \tag{4}$$

The main theme of this work is a bottom-up approach focusing on local subsystems' properties and their mutual interconnections from which we can deduce desired properties for the overall system. Hence, we first concentrate on the isolated subsystems, i.e., neglecting the coupling costs, and define the set of a subsystem's local optimal steady states as

$$\mathbb{Z}_i^* = \{(x_i^*, u_i^*) \in \mathbb{Z}_i^s \,|\, \ell_{ii}(x_i^*, u_i^*) \leq \ell_{ii}(x_i, u_i) \forall (x_i, u_i) \in \mathbb{Z}_i^s\}.$$

Moreover, we assume that each subsystem alone is (strictly) dissipative.

Definition 1 ([6], [12]): A control system (1) is dissipative with respect to a supply rate $s_i : \mathbb{Z}_i \to \mathbb{R}$ if there exists a non-negative function $\lambda_i : \mathbb{X}_i \to \mathbb{R}$, a positive semidefinite function $\rho_i : \mathbb{X}_i \to \mathbb{R}$, and $\bar{x}_i \in \mathbb{X}_i$ such that

$$\lambda_i(f_i(x_i, u_i)) - \lambda_i(x_i) \leq s_i(x_i, u_i) - \rho_i(x_i - \bar{x}_i) \tag{5}$$

for all $(x_i, u_i) \in \mathbb{Z}_i$. If there exists ρ_i positive definite, then the system is said to be strictly dissipative.

Assumption 1: Each subsystem $i \in \mathbb{I}_{[1,P]}$ is dissipative on $\mathbb{Z}_i$ with respect to the supply rate $s_i(x,u) := \ell_{ii}(x_i, u_i) - \ell_{ii}(x_i^s, u_i^s)$ for all $(x_i^s, u_i^s) \in \mathbb{Z}_i^*$, i.e., the following inequality holds for all $(x_i, u_i) \in \mathbb{Z}_i$

$$\begin{aligned} &\lambda_i(f_i(x_i, u_i)) - \lambda_i(x_i) \\ &\quad \leq \ell_{ii}(x_i, u_i) - \ell_{ii}(x_i^s, u_i^s) - \rho_i(x_i - x_i^s). \end{aligned} \tag{6}$$

This dissipativity assumption is explicitly formulated for the case of the optimal steady state (x_i^s, u_i^s) not being unique. Note that in the case of multiple optimal steady states, ρ_i can only be positive semidefinite.

Remark 2: Note that for local tracking costs, i.e., cost functions ℓ_{ii} that are positive definite w.r.t. (x_i^s, u_i^s), strict dissipativity is trivially fulfilled with $\lambda_i = 0$. Hence, all our following results on system interconnections also hold for local tracking costs as a special case of our more general setup considering arbitrary economic local cost functions.

As discussed in the introduction, the assumed (strict) dissipativity has strong implications on the optimal mode of operation of a system. Indeed, Assumption 1 is sufficient (and under some weak controllability conditions also necessary) for the optimal mode of operation of the subsystem being at the optimal steady state, cf. [15]. In an economic MPC context, this property immediately yields asymptotic averaged optimality of the closed loop. Consequently, in this work we investigate under which conditions the assumed local dissipativity properties carry over to the interconnected overall system, and hence, allow us to make statements about the optimal mode of operation of the overall system. Moreover, a resulting structured dissipativity property for the overall system can be exploited by non-iterative distributed economic MPC schemes as shown in [11].

In our previous work [11] it turned out that verification of strict dissipativity of the overall system was only possible in very special situations. Therefore, in this work we introduce the concept of approximate dissipativity and aim at verifying this relaxed version of (strict) dissipativity for the interconnected overall system.

Definition 3 (Approximate dissipativity): A control system as (1) is approximately dissipative with respect to a supply rate $s_i : \mathbb{Z}_i \to \mathbb{R}$ and suboptimality $\alpha_i \geq 0$ if there exists a non-negative function $\lambda_i : \mathbb{X}_i \to \mathbb{R}$, a positive semidefinite function $\rho_i : \mathbb{X}_i \to \mathbb{R}$, and $\bar{x}_i \in \mathbb{X}_i$ such that

$$\lambda_i(f_i(x_i,u_i)) - \lambda_i(x_i) \leq s_i(x_i,u_i) - \rho_i(x_i - \bar{x}_i) + \alpha_i \quad (7)$$

for all $(x_i,u_i) \in \mathbb{Z}_i$. If there exists ρ_i positive definite, then the system is said to be approximately strictly dissipative.

Note that this definition of approximate dissipativity reduces to classical dissipativity if the suboptimality is $\alpha_i = 0$. The concept of approximate dissipativity is discussed in Section III.

To summarize, the main goal of this work is to provide approximate dissipativity for the interconnected overall system (3) with overall cost function (4), i.e., verifying that there exists a non-negative function $\Lambda : \mathbb{X} \to \mathbb{R}$, $\bar{\rho} : \mathbb{X} \to \mathbb{R}$ positive semidefinite, $(\bar{x},\bar{u}) \in \mathbb{Z}^s$, and $\bar{\alpha} \geq 0$ such that

$$\Lambda(f(x,u)) - \Lambda(x) \leq \bar{s}(x,u) - \bar{\rho}(x - \bar{x}) + \bar{\alpha} \quad (8)$$

holds for all $(x,u) \in \mathbb{Z}$ with respect to the supply rate

$$\bar{s}(x,u) = \sum_{i=1}^{P} \ell_i(x_i,u_i,x_{-i}) - \ell_i(\bar{x}_i,\bar{u}_i,\bar{x}_{-i}). \quad (9)$$

III. Approximate dissipativity

In this section we discuss the notion of approximate dissipativity as introduced in Definition 3. The definition and this study of approximate dissipativity are independent of the above multi agent system setup, which is why we consider in this section a general control system of the form (3) with supply rate $s(x,u) = \ell(x,u) - \ell(\bar{x},\bar{u})$ and $(\bar{x},\bar{u}) \in \mathbb{Z}^s$ being a steady state of the system. Hence, for notational simplicity, in this section we refer to the above introduced (overall) system (3) with economic cost function $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$, but we would like to stress that the results presented in this section do not rely on any structure of f or ℓ. As briefly mentioned above, the notion of approximate dissipativity is useful in situations where we do not succeed in providing a (strict) dissipativity property for a given system and supply rate (9), but still would like to make statements about (sub)optimality of system operation with respect to a steady state $(\bar{x},\bar{u}) \in \mathbb{Z}^s$. For this reason, we chose the term "approximate dissipativity" (as opposed to "shortage" in the context of passivity [16, Chapter 6]), since it is indeed only an inexact dissipativity characterization owed to the fact that, e.g., the wrong storage function is employed, a suboptimal steady state is considered, or the optimal mode of system operation is actually not steady-state operation.

Proposition 4: Assume that the control system (3) is approximately dissipative with supply rate $s(x,u) = \ell(x,u) - \ell(\bar{x},\bar{u})$. Then for each feasible input sequence $u(\cdot)$ and associated state sequence $x(\cdot)$ it holds that

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} \geq \ell(\bar{x},\bar{u}) - \alpha.$$

Proof: The proof follows along the lines of the proof of [6, Proposition 6.4]. Due to λ being bounded from below, we obtain

$$\begin{aligned} 0 &\leq \liminf_{T\to\infty} \frac{1}{T}\left(\lambda(x(T)) - \lambda(x(0))\right) \\ &\leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} - \ell(\bar{x},\bar{u}) + \alpha. \end{aligned}$$

■

In other words, this performance result states that steady-state operation at $(\bar{x},\bar{u})$ is optimal up to the suboptimality α. Note, however, that this does not necessarily imply existence of any feasible system trajectory outperforming steady-state operation at $(\bar{x},\bar{u})$, e.g., in the case when the system is actually optimally operated at this steady state but an unsuitable storage function is considered; approximate dissipativity only bounds the *potential* suboptimality of steady-state operation.

Remark 5: For an approximately strictly dissipative system, the above derivation yields

$$\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \rho(x(k)-\bar{x})}{T} - \alpha \leq \liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \ell(x(k),u(k))}{T} - \ell(\bar{x},\bar{u}).$$

Hence, every system trajectory with $\liminf_{T\to\infty} \frac{\sum_{k=0}^{T-1} \rho(x(k)-\bar{x})}{T} > \alpha$ yields an averaged performance, which is worse than steady-state operation. This implies that the true optimal system trajectory resides on average inside the region $\{x \in \mathbb{X} \mid \rho(x-\bar{x}) \leq 2\alpha\}$.

Remark 6: In an economic MPC formulation with a terminal equality constraint requiring the predicted state trajectory to end at $\bar{x}$, approximate dissipativity directly yields a bound on the averaged performance of the MPC closed loop of $\ell(\bar{x},\bar{u}) + \alpha$ by slightly adapting the proof of [6, Theorem 1]. Moreover, practical (asymptotic) stability of $\bar{x}$ in the MPC closed loop follows, with the size of the convergence region being proportional to α, cf. [17].

In the multi agent scenario of the work at hand, it is obvious that considering an approximate dissipativity property makes sense, since we establish this property based on minimal assumptions on the local subsystems and their interconnection only. However, even in classical centralized economic MPC applications, approximate dissipativity may appear and provide useful insights. For example, verifying (strict) dissipativity for a given system is in general a very hard problem. The recent work [18], for example, employed sum-of-squares methods in order to find a storage function fulfilling the dissipation inequality. Numerical issues may prevent that this algorithm finds an exact solution in some cases, however, an approximate dissipativity result could still be obtained. For systems that are optimally operated at a periodic orbit, the situation is even worse, since besides (parameterized or multiple) storage functions, even the optimal period length has to be determined. In some applications, however, an approximate performance result through approximate dissipativity with respect to a steady state might be satisfactory. Finally, we would like to mention that a similar notion to approximate dissipativity appeared in the context of system dynamics subject to additive disturbances as robust dissipativity, see [19]. This definition looks similar to ours, however, there it has been considered as an exact dissipativity property for any possible realization of the disturbance, and hence, our viewpoint and interpretation is conceptually different.

IV. Structured approximate dissipativity of the overall system

In this section we approach the main problem stated above, which is to establish approximate dissipativity for the interconnected overall system based on the subsystems' local dissipativity and their interconnection structure. These givens suggest to simply choose the sum of the local storage functions as the storage function for the overall system, i.e., $\Lambda(x) = \sum_{i=1}^{P} \lambda_i(x_i)$. Hence, in order to verify approximate dissipativity of the overall system, we need to provide $\bar{\rho} \succeq 0$, $\bar{\alpha} \geq 0$ and $(\bar{x}, \bar{u}) \in \mathbb{Z}^s$ that satisfy the following inequality

$$\sum_{i=1}^{P} \Big(-\ell_{ii}(x_i^s, u_i^s) - \rho_i(x_i - x_i^s) \Big) \leq \sum_{i=1}^{P} \Big(-\ell_{ii}(\bar{x}_i, \bar{u}_i) + \sum_{j \in \mathcal{N}_i} \ell_{ij}(x_i, x_j) - \ell_{ij}(\bar{x}_i, \bar{x}_j) \Big) - \bar{\rho}(x - \bar{x}) + \bar{\alpha} \tag{10}$$

for all $x \in \mathbb{X}$.

Lemma 7: Let Assumption 1 hold and let $(\bar{x}, \bar{u}) \in \mathbb{Z}^s$ be a steady state of the overall system (3). If there exists a positive semidefinite function $\bar{\rho} : \mathbb{X} \to \mathbb{R}$ such that (10) is satisfied for all $(x, u) \in \mathbb{Z}$, then the overall system (3) is approximately dissipative with respect to the supply rate $\bar{s}(x, u) := \sum_{i=1}^{P} \ell_i(x_i, u_i, x_{-i}) - \ell_i(\bar{x}_i, \bar{u}_i, \bar{x}_{-i})$ and suboptimality $\bar{\alpha}$. If there exists $\bar{\rho}$ positive definite, then the overall system is approximately strictly dissipative.

Proof: This directly follows from using $\Lambda(x) = \sum_{i=1}^{P} \lambda_i(x_i)$ as storage function together with (10). ∎

Note that this is still a challenging problem, however it is potentially significantly simpler than directly finding a suitable storage function $\Lambda(x)$ which yields approximate dissipativity of the overall system. Moreover, for certain classes of systems and interconnection cost functions, as discussed below, we can provide simple explicit solutions. Obviously, the free variables $\bar{\rho}$, $(\bar{x}, \bar{u})$ and $\bar{\alpha}$ in (10) could be found by solving an optimization problem, e.g., minimizing the suboptimality $\bar{\alpha}$, or minimizing the region $\{x \in \mathbb{X} \,|\, \bar{\rho}(x - \bar{x}) \leq 2\bar{\alpha}\}$. The result above is rather general and does not yield much insight into the problem, which is why we specialize the system setup to the case of diffusive quadratic coupling costs in the following.

A. Diffusive quadratic coupling costs

In this section, we specialize the coupling costs between the subsystems to be of the following quadratic form.

Assumption 2: Let $\ell_{ij}(x_i, x_j) = q_{ij}(x_i - x_j)^\top (x_i - x_j)$, where q_{ij} is given as the weight of the (i, j)-edge of G.

Such diffusive coupling costs appear in many practical applications where, e.g., some synchronization or agreement between the subsystems is desired. Considering diffusive quadratic coupling costs defined by the interconnection graph and its weights allows us to rewrite the overall system's cost using a compact graph theoretic notation

$$\ell(x, u) = \sum_{i=1}^{P} \ell_{ii}(x_i, u_i) + x^\top (L \otimes I_n) x.$$

Here, $L \in \mathbb{R}^{P \times P}$ is the weighted undirected Laplacian matrix of the interconnection graph G given as $L = BWB^\top$ with B the incidence matrix of G and $W \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ the diagonal matrix of weights with entries $W_{ee} = q_{ij}$ if vertex i is the tail and vertex j is the head of edge e. For the graph G being weakly connected, L is positive semidefinite, with a single 0 eigenvalue and according eigenvector $\mathbf{1}_P$ [14].

Assumption 3: The graph G is weakly connected.

In the case of a graph consisting of several disconnected components, all our following results hold for each of the separate subgraphs.

Aside from the coupling costs, we also assume the local subsystems' strict dissipativity to hold quadratically. Note that this implies no general restrictions on the class of subsystems and local economic cost functions, which are both still considered arbitrary nonlinear.

Assumption 4: Let Assumption 1 hold for

$$\rho_i(x_i - x_i^s) = q_{ii}(x_i - x_i^s)^\top (x_i - x_i^s) \tag{11}$$

with $q_{ii} \geq 0$ for all $i \in \mathbb{I}_{[1,P]}$.

Revisiting the above condition (10) for approximate dissipativity of the overall system, it is seen that its right-hand side becomes quadratic, i.e., $\sum_{i=1}^{P} \rho_i(x_i - x_i^s) + \sum_{j \in \mathcal{N}_i} \ell_{ij}(x_i - x_j) = (x - x^s)^\top Q_n (x - x^s) + x^\top L_n x$ with $L_n = L \otimes I_n$, $Q_n = Q \otimes I_n$, and $Q := \operatorname{diag}(q_{11}, \ldots, q_{PP})$. For $Q_n \neq 0$, i.e., at least one subsystem is strictly dissipative, we may combine these terms to arrive at $(x - x^s)^\top Q_n (x - x^s) + x^\top L_n x = (x - \tilde{x})^\top (Q_n + L_n)(x - \tilde{x}) + \tilde{c}$, with $\tilde{x} = (Q_n + L_n)^{-1} Q_n x^s$ and $\tilde{c} = x^{s\top} Q_n x^s - x^{s\top} Q_n \tilde{x}$ by completing the square. Observe that $(Q_n + L_n)^{-1} = ((Q + L) \otimes I_n)^{-1}$ is indeed invertible since $Q + L$ is positive definite: Positive definiteness of $Q + L$ results from L being positive semidefinite with $\ker L = \mathbf{1}_P$ and $Q \succeq 0$ being diagonal with at least one strictly positive diagonal element q_{ii} and $\mathbf{1}_P^\top Q \mathbf{1}_P > 0$. In the case of $Q_n = 0$, we may choose $\tilde{x} \in \operatorname{span}\{\mathbf{1}_P \otimes v\}$ for an arbitrary $v \in \mathbb{R}^n$, and $\tilde{c} = 0$. The point $\tilde{x}$ can be seen as a "compromise point", which trades off the subsystems' individual preferences represented by their local strict dissipativity property against the synchronization requirement. Eventually, for the considered case of quadratic coupling costs, verification of approximate dissipativity of the overall system reduces to finding $\bar{\rho} \succeq 0$,

$(\bar{x},\bar{u}) \in \mathbb{Z}$, and $\bar{\alpha} \geq 0$ such that

$$\sum_{i=1}^{P} \Big(\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s)\Big) + \bar{x}^\top L_n \bar{x} \qquad (12)$$
$$+ \bar{\rho}(x-\bar{x}) - \bar{\alpha} \leq (x-\tilde{x})^\top (Q_n + L_n)(x-\tilde{x}) + \tilde{c}$$

holds for all $x \in \mathbb{X}$. Thus, the problem reduces to finding a possibly quadratic function $\bar{\rho}$ centered at $\bar{x}$, which lower bounds $(x-\tilde{x})^\top(Q_n+L_n)(x-\tilde{x})$ subject to the remaining constant terms, which can always trivially be fulfilled for $\bar{\rho}=0$.

Lemma 8: Let Assumptions 2–4 hold. The overall system (3) is approximately (strictly) dissipative with supply rate $\bar{s}(x,u) := \sum_{i=1}^P \ell_i(x_i,u_i,x_{-i}) - \ell_i(\bar{x}_i,\bar{u}_i,\bar{x}_{-i})$ for any $(\bar{x},\bar{u}) \in \mathbb{Z}^s$ (if at least one subsystem is strictly dissipative).

Proof: The claim follows from the above derivations. In particular, for approximate dissipativity, $\bar{\rho} := 0$ is a possible choice with the corresponding suboptimality $\bar{\alpha} = \sum_{i=1}^P \ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s) + \bar{x}^\top L_n\bar{x} - \tilde{c}$. For strict dissipativity, thanks to Q_n+L_n being positive definite, there exists $\bar{Q} \in \mathbb{R}^{nP\times nP}$ such that $(Q_n+L_n) - \bar{Q} \succ 0$, and we may choose $\bar{\rho}(x-\bar{x}) := (x-\bar{x})^\top \bar{Q}(x-\bar{x})$. In this case, the suboptimality increases to $\bar{\alpha} = \sum_{i=1}^P \ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s) + \bar{x}^\top L_n \bar{x} - \tilde{c} + \Delta c$ with $\Delta c = (\bar{Q}\bar{x} - (Q_n+L_n)\tilde{x})^\top (Q_n + L_n - \bar{Q})^{-1}(\bar{Q}\bar{x} - (Q_n+L_n)\tilde{x}) - \tilde{x}^\top(Q_n+L_n)\tilde{x} + \bar{x}^\top\bar{Q}\bar{x}$, which follows again by completing the squares and ensures that $\bar{\rho}(x-\bar{x}) - \Delta c$ bounds $(x-\tilde{x})^\top(Q_n+L_n)(x-\tilde{x})$ from below. ■

Remark 9: This result highlights the main effect of considering the concept of approximate dissipativity in this work. Whereas in [11] we could only make statements on dissipativity of the overall system under special conditions, here we provide approximate dissipativity for any system interconnection of the considered kind.

A number of interesting explicit results directly follow for specific scenarios. First, we recover the intuitive result of [11, Lemma 9] that if all subsystems share the same optimal steady state, then strict dissipativity of the overall system directly follows.

Proposition 10: Let Assumptions 2–4 hold, let at least one subsystem $i \in \mathbb{I}_{[1,P]}$ be strictly dissipative, and let $x^s = \mathbf{1}_P \otimes x_i^s$. Then the overall system is strictly dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(x^s,u^s)$. Moreover, (x^s,u^s) is the optimal steady state of the overall system.

Proof: The claim is proved by verifying (12) for $\bar{\alpha}=0$. First, note that $\tilde{x} = (Q_n+L_n)^{-1}Q_n x^s = x^s = \mathbf{1}_P \otimes x_1^s$, since $Q_n(\mathbf{1}_P \otimes x_1^s) = (Q_n+L_n)(\mathbf{1}_P\otimes x_1^s)$ by $(L\otimes I_n)(\mathbf{1}_P \otimes x_1^s) = (L\mathbf{1}_P)\otimes(I_n x_1^s) = 0$, and $\tilde{c}=0$. Consequently, for $\bar{x} = x^s$ and Q_n+L_n being positive definite, there exists $\bar{Q} \succ 0$ such that $(x-\tilde{x})^\top(Q_n+L_n)(x-\tilde{x}) \geq (x-\bar{x})^\top\bar{Q}(x-\bar{x}) =: \bar{\rho}(x-\bar{x})$, and hence (12) is satisfied with this choice of $\bar{\rho}$ and $\bar{\alpha}=0$. ■

Next, we investigate the case of the compromise point $\tilde{x} = (Q_n+L_n)^{-1}Q_n x^s$ being a steady state of the overall system.

Proposition 11: Let Assumptions 2–4 hold, let at least one subsystem be strictly dissipative, and assume $\exists \bar{u} \in \mathbb{U}$: $(\bar{x},\bar{u}) \in \mathbb{Z}^s$ for $\bar{x} = \tilde{x} = (Q_n+L_n)^{-1}x^s$. Then the overall system is approximately strictly dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(\bar{x},\bar{u})$ and suboptimality $\bar{\alpha} = \sum_{i=1}^P \big(\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s)\big) - (\bar{x}-x^s)^\top Q_n(\bar{x}-x^s)$.

Proof: The claim is proved by verifying (12). A possible choice is $\bar{\rho}(x-\bar{x}) = (x-\bar{x})^\top(Q_n+L_n)(x-\bar{x})$ which allows us to set $\bar{\alpha} = -\tilde{c} + \sum_{i=1}^P \big(\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s)\big) + \bar{x}^\top L_n\bar{x} = \sum_{i=1}^P \big(\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s)\big) - (\bar{x}-x^s)^\top Q_n(\bar{x}-x^s)$. ■

This result highlights some peculiarities of the approach taken in this work. In the calculation of the compromise point, information about the subsystems' individual local cost functions is only considered by means of the quadratic bound given by the subsystems' dissipativity property. As a consequence, even in the ideal case of the compromise point $\tilde{x}$ being a steady state of the overall system, the suboptimality $\bar{\alpha}$ is "as large as the looseness of the local strict dissipativity bound". This insight sheds some light on our result in [11], where we need to employ a restrictive "tightness" assumption on the local economic cost functions and, accordingly, the dissipativity bound in order to establish strict dissipativity for the overall system.

Corollary 12: Let the conditions of Proposition 11 hold, and let $\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s) = q_{ii}(\bar{x}_i - x_i^s)^\top(\bar{x}_i - x_i^s)$ for all $i \in \mathbb{I}_{[1,P]}$. Then the overall system is strictly dissipative with respect to the supply rate $s(x,u) = \ell(x,u) - \ell(\bar{x},\bar{u})$.

Proof: The claim follows from Proposition 11 by noting that now $\bar{\alpha} = \sum_{i=1}^P \big(\ell_{ii}(\bar{x}_i,\bar{u}_i) - \ell_{ii}(x_i^s,u_i^s)\big) - (\bar{x} - x^s)^\top Q_n(\bar{x}-x^s) = 0$, i.e., the suboptimality vanishes. ■

The calculation of the compromise point $\tilde{x}$ is based on minimal knowledge of the subsystems, namely the local subsystems' preference in terms of their optimal steady states and the strictness of their local dissipativity only. However, global information is required, since despite Q_n+L_n is structured, its inverse is in general a full matrix. Even though the compromise point $\tilde{x}$ could be computed by efficient and well-established distributed optimization algorithms, we mainly consider the above approximate dissipativity results useful as analysis tool for making statements on the suboptimality of a particular steady state, or to exploit the resulting structured (approximate) dissipativity property in the analysis of non-iterative distributed economic MPC algorithms, similarly as shown in [11, Theorem 12]. On the contrary, in the following section we show how the concept of approximate dissipativity can constructively be utilized for the estimation of induced suboptimality when adding a new subsystem to the network.

V. Online addition of a new subsystem

This section showcases how the concept of approximate dissipativity can constructively be employed in the exemplary scenario of adding an additional subsystem to the network. In general, addition of a new subsystem to the network will change the overall system's approximate dissipativity property. Following the main theme of this work, we would like to assess the influence of adding one subsystem to the network based on local information available to the newly added system. Hence, we avoid any overall recomputations (and in an MPC context also reconfiguration), e.g., of the compromise point, while still giving an estimate for the potential degradation of overall performance (by means of an increase of the suboptimality in the overall system's approximate dissipativity).

Proposition 13: Consider an approximately dissipative overall system composed of P subsystems with supply rate $s(x,u) = \ell(x,u) - \ell(\bar{x},\bar{u})$ and suboptimality α and consider positive coupling costs $\ell_{ij}(x_i,x_j) \geq 0$ for all $x_i \in \mathbb{X}_i, x_j \in \mathbb{X}_j$. Suppose that one additional local dissipative subsystem $i = 0$ with supply rate $s_0(x_0,u_0) = \ell_{00}(x_0,u_0) - \ell(x_0^s,u_0^s)$ is added to the network, i.e., the node 0 and a set of adjacent edges $\mathcal{E}_0$ with weights $\mathcal{W}_0$ to the interconnection graph G. Then the extended overall system retains approximate dissipativity with an increase in the suboptimality of $\Delta\alpha = \sum_{j:(0,j)\in\mathcal{E}_0} \ell_{0j}(x_0^s,\bar{x}_j) + \sum_{j:(j,0)\in\mathcal{E}_0} \ell_{j0}(\bar{x}_j,x_0^s)$.

Proof: The main idea is to render the addition of the new subsystem somehow "neutral" with respect to the approximate dissipativity of the overall system. Hence, we keep the reference point $\bar{x}$ of the previous overall system unaltered to refrain from overall computations based on information of all subsystems, and to directly build the approximate dissipativity result of the enlarged overall system upon the previous approximate dissipativity property.

$$\begin{aligned}
&\sum_{i=0}^{P} \lambda_i(f_i(x_i,u_i)) - \lambda(x_i) \leq \sum_{i=1}^{P} \Big(\ell_{ii}(x_i,u_i) - \ell_{ii}(\bar{x}_i,\bar{u}_i) \\
&\quad + \sum_{j\in\mathcal{N}_i} \ell_{ij}(x_i,x_j) - \ell_{ij}(\bar{x}_i,\bar{x}_j) \Big) - \bar{\rho}(x-\bar{x}) + \alpha \\
&\quad + \ell_{00}(x_0,u_0) - \ell_{00}(x_0^s,u_0^s) \\
&\leq \sum_{i=1}^{P} \Big(\ell_{ii}(x_i,u_i) - \ell_{ii}(\bar{x}_i,\bar{u}_i) \Big) + \ell_{00}(x_0,u_0) - \ell_{00}(x_0^s,u_0^s) \\
&\quad + \sum_{(i,j)\in\mathcal{E}} \Big(\ell_{ij}(x_i,x_j) - \ell_{ij}(\bar{x}_i,\bar{x}_j) \Big) + \sum_{(i,j)\in\mathcal{E}_0} \Big(\ell_{ij}(x_i,x_j) \\
&\quad - \sum_{j:(0,j)\in\mathcal{E}_0} \ell_{0j}(x_0^s,\bar{x}_j) - \sum_{j:(j,0)\in\mathcal{E}_0} \ell_{j0}(\bar{x}_j,x_0^s) - \rho^*\big(\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_0^s \\ \bar{x} \end{smallmatrix}\right] \big) + \alpha^*.
\end{aligned}$$

Hence we arrive at approximate dissipativity of the extended overall system with a possible choice of $\rho^*(\left[\begin{smallmatrix} x_0 \\ x \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_0^s \\ \bar{x} \end{smallmatrix}\right]) = \bar{\rho}\big(x - \bar{x}\big)$ and suboptimality $\alpha^* = \alpha + \sum_{j:(0,j)\in\mathcal{E}_0} \ell_{0j}(x_0^s,\bar{x}_j) + \sum_{j:(j,0)\in\mathcal{E}_0} \ell_{j0}(\bar{x}_j,x_0^s)$. ■

Remark 14: An interpretation of the above result, which becomes also visible in the proof, is that we shift all induced suboptimality to the newly added local subsystem in terms of neglecting its local preference in terms of (strict) local dissipativity. Thereby, we leave the approximate dissipativity result for the existing overall system unaltered and capture all influence of the newly added system in $\Delta\alpha$. Note that the calculation of $\Delta\alpha$ can indeed be carried out locally by the newly added subsystem. This, however, comes at the price of a conservative suboptimality for the extended overall system, which is usually higher than considering the extended overall system as a whole and following any of the approaches from Section IV. More elaborate strategies diminishing this effect are subject to future work.

VI. Conclusions

In this work, we considered a set of dynamically decoupled, locally dissipative subsystems interconnected through coupling costs. We followed a bottom-up approach establishing dissipativity of the interconnected overall system from the subsystems' local dissipativity and their interconnection structure. Introducing the concept of approximate dissipativity enables us to provide this relaxed property for any system of the considered class. For approximately dissipative systems, the performance and stability results derived from classical dissipativity directly translate to suboptimality estimates and practical stability, respectively. Moreover, we showed how this property can constructively be utilized in a plug-and-play scenario.

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