Economic model predictive control with self-tuning terminal cost

Matthias A. Müller, David Angeli, and Frank Allgöwer

Abstract—In this paper, we propose an economic model predictive control (MPC) framework with a self-tuning terminal weight, which builds on a recently proposed MPC algorithm with a generalized terminal state constraint. First, given a general time-varying terminal weight, we derive an upper bound on the closed-loop average performance which depends on the limit value of the predicted terminal state. After that, we derive conditions for a self-tuning terminal weight such that bounds for this limit value can be obtained. Finally, we propose several update rules for the self-tuning terminal weight and analyze their respective properties. We illustrate our findings with several examples.

I. INTRODUCTION

Model predictive control has become a very popular and successful control strategy in recent years, which is based on the repeated solution of a finite horizon optimal control problem. So far, the most commonly used MPC formulation is that of a tracking problem, meaning that the involved cost function is assumed to be positive definite with respect to a certain setpoint or trajectory to be tracked (see, e.g., [2]). However, this basic assumption need not be satisfied in general, which in particular is the case when optimizing the economics of a process (for a recent example, see, e.g., [3]). In order to overcome this limitation, an economic MPC formulation has recently been proposed [4], where this assumption is not made but a general cost function can be used. Within such an economic MPC framework, different properties of the resulting closed-loop system have been studied such as average performance or convergence to the optimal steady-state. To this end, different assumptions and variants of economic MPC have been used, including terminal constraints [4–7], Lyapunov-like constraints [8], and certain controllability [9] and dissipativity [4, 6, 9–11] conditions.

In this paper we consider an economic MPC setup involving a generalized terminal state constraint, meaning that the endpoint of the predicted state sequence has to be equal to some arbitrary steady-state in contrast to the optimal one as in [4, 5]. Such a setup has been proposed in the context of tracking MPC (see, e.g., [12–15]) and recently also within an economic MPC framework [14–17]. The big advantage of such a generalized terminal constraint is that a possibly much larger region of attraction is obtained, and a loss of feasibility can be prevented which otherwise might occur if the cost function (and hence the optimal steady-state) changes online. In particular, in [16, 17], the authors propose to use a slightly modified economic cost function and in addition an offset cost function which penalizes the distance of the predicted terminal steady-state \( x(N|t) \) to the optimal steady-state \( x^* \). On the other hand, in [14, 15] the authors use the original (economic) cost function and in addition a terminal cost term which penalizes the economic cost of the predicted terminal steady-state \( x(N|t) \). It is shown that if the terminal weight is large enough, then the cost of the predicted terminal steady-state will be arbitrarily close to the cost of the best reachable steady-state. Furthermore, under additional assumptions and by further modifying the MPC algorithm, i.e., if necessary, following the previously optimal solution, the cost of the predicted terminal steady-state, and hence also the average performance of the closed-loop system, will eventually be arbitrarily close to the cost of the best overall steady-state [15].

In this work, we develop an economic MPC algorithm with a self-tuning terminal cost, using a setup similar to [14]. The advantage of such a self-tuning terminal weight in comparison to a fixed terminal weight is that we do not necessarily need to make the terminal weight large in order to guarantee certain performance properties. This is desirable both from a numerical point of view as well as in order not to modify the original economic cost function too much. In fact, we illustrate with a simple example that in some cases, a smaller terminal weight leads to a better closed-loop average performance than a large terminal weight. Furthermore, we want to give (average) performance guarantees for the closed-loop system without further modifying the MPC algorithm as in [15]. The remainder of this paper is organized as follows. In Section II, some preliminaries and the precise problem statement are introduced; furthermore, we illustrate by means of an example that larger values of the terminal weight do not necessarily lead to a better closed-loop average performance. Section III then gives an upper bound for the closed-loop average performance when using a general time-varying terminal weight. In Section IV, we consider certain self-tuning update rules for the terminal cost and show that the cost of the predicted terminal steady-state (and hence also the closed-loop average performance) can be upper bounded by the best "robustly" achievable steady-state cost of the \( \omega - \)
Section VII concludes the paper.

Finally, let

\[ x(t+1) = f(x(t), u(t)), \quad x(0) = x_0, \]

with \( t \in \mathbb{N}_0 \), where \( x \in \mathbb{X} \subseteq \mathbb{R}^n \), \( u \in \mathbb{U} \subseteq \mathbb{R}^m \), and \( f \) is assumed to be continuous. The system is subject to (possibly coupled) state and input constraints \((x, u) \in \mathbb{Z}\) for some compact set \( \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U} \). Denote by \( \mathbb{Z}_X \) the projection of \( \mathbb{Z} \) on \( \mathbb{X} \). Define the set of steady-states which are reachable in \( N > 0 \) steps from a point \( y \in \mathbb{Z}_X \) as

\[ \mathcal{X}_N(y) := \{ x \in \mathbb{Z}_X : \exists u \in \mathbb{U}^{(N+1)} \text{ s.t. } x(0) = y, \]

\[ x(j+1) = f(x(j), u(j)) \forall j \in [0, N-1], x(N) = x, \]

\[ x = f(x, u(N), (x(j), u(j))) \in \mathbb{Z} \forall j \in [0, N]. \]

(2)

Note that for each \( y \in \mathbb{Z}_X \), the set \( \mathcal{X}_N(y) \) is compact as \( \mathbb{Z} \) is compact and \( f \) is continuous. System (1) is equipped with a stage cost \( \ell : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R} \), which is assumed to be continuous. Denote the best achievable steady-state cost from a point \( y \in \mathbb{Z}_X \) by

\[ \ell_{min}(y) := \min_{x, u} \ell(x, u) \]

\[ \text{s.t. } x \in \mathcal{X}_N(y), \quad (x, u) \in \mathbb{Z}, \]

\[ x = f(x, u), \]

(3)

Furthermore, in the following, we need the notion of the best 

\[ \ell_{min}(y, \varepsilon) := \sup_{x \in \mathbb{Z}_X} \ell_{min}(z) \]

the supremum of the best achievable steady-state cost from a point \( y \in \mathbb{Z}_X \), which we define as follows. For each \( \varepsilon \geq 0 \), denote by

\[ \ell_{min}(y, \varepsilon) := \sup_{x \in \mathbb{Z}_X} \ell_{min}(z) \]

(4)

(5)

Note that the limit in (5) exists as \( \ell_{min}(y, \varepsilon) \) is monotonically decreasing when \( \varepsilon \) decreases to zero. From the definitions in (3) and (5), it immediately follows that for each \( y \in \mathbb{Z}_X \) we have \( \ell_{min}(y) \leq \ell_{min}(x) \); however, equality does in general not hold as \( \ell_{min}(y, \varepsilon) \) is not necessarily continuous in \( \varepsilon \) at \( \varepsilon = 0 \) (for a simple example of this fact, see Example 4 in Section VI). Finally, let \((x_s, u_s)\) denote an overall optimal steady-state, i.e., \((x_s, u_s)\) satisfies

\[ \ell(x_s, u_s) = \min_{x(u) \in \mathbb{Z}_X} \ell(x, u); \]

(6)

Note that without loss of generality, we can assume that \( \ell(x_s, u_s) = 0 \).

Now consider the following economic MPC algorithm, which is a variation of the one introduced in [14]. Namely, at each time \( t \) with \( x := x(t) \), the following optimization problem is solved:

\[ \min_{u} J(x, u, \beta) = \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + \beta(t) \ell(x(N|t), u(N|t)) \]

(7)

subject to

\[ x(0|t) = x \]

\[ x(k+1|t) = f(x(k|t), u(k|t)) \quad k \in [0, N-1] \]

\[ x(N|t) = f(x(N|t), u(N|t)) \]

\[ \ell(x(N|t), u(N|t)) \leq \lambda(t). \]

(8a)

(8b)

(8c)

(8d)

(8e)

for some possibly time-varying terminal weight \( \beta \) and \( \lambda \) specified in the following. As pointed out in the Introduction, the special feature of the above optimization problem is the generalized terminal constraint in (8d), meaning that the predicted terminal state \( x(N|t) \) has to be equal to an arbitrary steady-state and not to a specific one. Denote the optimal solution to problem (7)–(8) by \( u^* := [u^*(0|t)^T, \ldots, u^*(N|t)^T]^T \) and the corresponding state sequence by \( x^* := [x^*(0|t)^T, \ldots, x^*(N|t)^T]^T \). As usual in MPC, the first part of the optimal input sequence, \( u^*(0|t) \), is applied to the system at time \( t \). The optimal value function is denoted by \( V(x, \beta) := J(x, u^*, \beta) \), which depends on the terminal weight \( \beta \). The parameter \( \lambda \) is updated according to the cost of the previous terminal state, i.e., the following closed-loop system is obtained:

\[ x(t+1) = f(x(t), u^*(0|t)) \]

\[ \lambda(t+1) = \ell(x^*(N|t), u^*(N|t)) \]

(9)

with an appropriate initialization of \( \lambda \) (we will comment more on this later in Section V). From (8d)–(9), it follows that the sequence \( \lambda(t) \) is nonincreasing and bounded from below (by \( 0 = \ell(x_s, u_s) \)), hence it converges. Denote the limit by \( \lambda_\infty := \lim_{t \to \infty} \lambda(t) \geq 0 \). Note that the sequence \( \lambda(t) \) is convergent irrespective of the evolution of the terminal weight \( \beta \), however, the limit \( \lambda_\infty \) does in general depend on \( \beta \).

As described in the Introduction, in the following we want to analyze the behavior of the closed-loop system (9) when using a self-tuning, time-varying terminal weight \( \beta \) which is not unnecessarily large, and without further modifying the MPC algorithm as in [15, Algorithm 3]. Before doing so, we shortly motivate our research with a simple example showing that larger values of \( \beta \) do not necessarily lead to smaller values of \( \lambda_\infty \) and to a better average performance of the closed-loop system.

1For simplicity, we assume that \( u^* \) is unique. If this is not the case, just assign a unique constant selection map to select one of the multiple minima.
Example 1: Consider the system $x(t+1) = x(t)u(t)$ with state and input constraint set $Z = \mathbb{X} \times \mathbb{U}$ with $\mathbb{U} = [-1.2, 1.2]$ and $\mathbb{X} = [-5, 5]$, cost function

$$\ell(x, u) = \frac{1}{4}x^4 - \frac{3}{4}x^3 + \frac{3}{2}x^2 + \frac{9}{4} + (u - 1)^2$$

and prediction horizon $N = 1$. The cost function $\ell$ is plotted in Figure 1(a) for $u = 1$. Figure 1(b) shows closed-loop state sequences with $x_0 = 1.2$ and four different constant values of $\beta$. As can be seen, $x$ converges to 0 and hence $\lambda_\infty = \ell(0, 1) = 9/4$ for both $\beta \equiv 1.5$ and $\beta \equiv 5$ (the same also happens for all larger values of $\beta$), whereas $x$ converges to 3 and hence $\lambda_\infty = \ell(3, 1) = 0$ for both $\beta \equiv 0.1$ and $\beta \equiv 1$. The reason for this is that for sufficiently large values of $\beta$, the control action is chosen such that the cost of the predicted terminal steady-state $x(1|t)$ becomes small. As can be seen from Figure 1(a), the best steady-state which can be reached within one step (given the above input constraints) is $x = 0$, and hence $u$ is chosen such that $x$ decreases. On the other hand, for small values of $\beta$, the terminal steady-state is not weighed as much, which results in an increasing $x$ and a better average performance of the resulting closed-loop system.

III. AVERAGE PERFORMANCE WITH TIME-VARYING TERMINAL WEIGHT

In this section, we study the average performance of the closed-loop system (9) when the terminal weight $\beta$ in (7) is time-varying. In order to prove certain bounds on the closed-loop average performance, we consider the following two assumptions on $\beta$. Later, in Section IV-B, we propose several update rules $B$ for the terminal weight $\beta$ such that these assumptions are satisfied. Now let $\gamma(t) := \beta(t) - \beta(t)$. Assumption 1: The sequence $\beta(t)$ satisfies $\gamma(t) \leq c$ and $\beta(t) \geq \beta$ for all $t \in \mathbb{I}_{\geq 0}$ and some constants $c, \beta \in \mathbb{R}$, and $\limsup_{t \to \infty} \gamma(t) \leq 0$.

Assumption 2: The sequence $\beta(t)$ satisfies $\gamma(t) \leq c$ and $\beta(t) \geq \beta$ for all $t \in \mathbb{I}_{\geq 0}$ and some constants $c, \beta \in \mathbb{R}$, and $\liminf_{t \to \infty} \beta(t) < \infty$.

Theorem 1: Assume that the optimization problem (7)--(8) is feasible at $t = 0$. Then it is feasible for all $t \in \mathbb{I}_{\geq 0}$. Consider the closed-loop system (9). If $\beta(t)$ satisfies Assumption 1, then

$$\limsup_{T \to \infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T} \leq \lambda_\infty.$$  

If $\beta(t)$ satisfies Assumption 2, then

$$\liminf_{T \to \infty} \frac{\sum_{t=0}^{T-1} \ell(x(t), u(t))}{T} \leq \lambda_\infty.$$  

Remark 1: Note that a constant $\beta$ trivially satisfies both Assumptions 1 and 2. The motivation for studying more complex, time-varying terminal weights $\beta$ is that in this case, statements about the value of $\lambda_\infty$, and hence, by Theorem 1, also on the average performance of the closed-loop system, can be made (see Section IV-A).

Proof of Theorem 1: As usual in MPC, a feasible solution to problem (7)--(8) at time $t + 1$ is given by the endpiece of the previously optimal solution appended by the steady-state input, i.e., $u(t+1) := [u^*(1|t)]^T, \ldots, u^*(N|t)^T, u^*(N|t)^T]^T$ (see, e.g., [2]). This means that the optimization problem (7)--(8) is recursively feasible. With this, we obtain

$$V(x(t+1), \beta(t+1)) - V(x(t), \beta(t)) \leq J(x(t+1), u(t+1), \beta(t+1)) - J(x(t), u^*(t), \beta(t)) = (1 + \gamma(t))\ell(x^*(N|t), u^*(N|t)) - \ell(x(t), u(t))$$

As discussed above, the sequence $\ell(x^*(N|t), u^*(N|t))$ is non-increasing in $t$ and converges to $\lambda_\infty$ for $t \to \infty$. This means that $\varepsilon(t) := \ell(x^*(N|t), u^*(N|t)) - \lambda_\infty$ converges to zero for $t \to \infty$. Furthermore, $0 \leq \varepsilon(t) \leq \varepsilon(0) < \infty$ for all $t \in \mathbb{I}_{\geq 0}$, where the last inequality follows from continuity of $\ell$ and compactness of $Z$. Summing the inequality in (12), for each $T \geq 1$ we obtain

$$V(x(T), \beta(T)) - V(x_0, \beta_0) \leq \sum_{t=0}^{T-1} \big( (1 + \gamma(t))\lambda_\infty + \varepsilon(t) \big) - \ell(x(t), u(t)).$$  

Now first consider the case where Assumption 1 is satisfied. Taking averages on both sides of (13), we obtain

$$\liminf_{T \to \infty} \frac{1}{T}(V(x(T), \beta(T)) - V(x_0, \beta_0)) \leq \liminf_{T \to \infty} \frac{1}{T} \left( \lambda_\infty + \sum_{t=0}^{T-1} \left[ \gamma(t)\lambda_\infty + (1 + \gamma(t))\varepsilon(t) - \ell(x(t), u(t)) \right] \right) \leq \lambda_\infty + \liminf_{T \to \infty} \frac{1}{T} \left( \sum_{t=0}^{T-1} \ell(x(t), u(t)) \right) + \limsup_{T \to \infty} \frac{1}{T} \left( \sum_{t=0}^{T-1} [\gamma(t)\lambda_\infty + (1 + \gamma(t))\varepsilon(t)] \right) \leq \lambda_\infty - \limsup_{T \to \infty} \frac{1}{T} \left( \sum_{t=0}^{T-1} \ell(x(t), u(t)) \right),$$  

where the last inequality follows from the fact that $\lim_{t \to \infty} \varepsilon(t) = 0$, $\limsup_{t \to \infty} \gamma(t) \leq 0$, $\lambda_\infty \geq 0$, and $0 \leq \varepsilon(t) \leq \varepsilon(0) < \infty$ and $\gamma(t) \leq c < \infty$ for all
for all $t \in \mathbb{I}_{\geq 0}$. On the other hand, as $\beta(t) \geq \beta$ for all $t \in \mathbb{I}_{\geq 0}$ and furthermore $\ell$ is continuous and $\mathbb{Z}$ compact, by definition of the cost function $J$ in (7) there exists a constant $\check{V} \in \mathbb{R}$ such that $V(x(t), \beta(t)) \geq \check{V}$ for all $t \in \mathbb{I}_{\geq 0}$. Hence we obtain

$$
\liminf_{T \to \infty} \frac{1}{T} \int_0^T \left( V(x(t), \beta(t)) - V(x_0, \beta_0) \right) dt \geq \liminf_{T \to \infty} \frac{1}{T} \int_0^T V(x(t), \beta(t)) dt + \liminf_{T \to \infty} \frac{-V(x_0, \beta_0)}{T} 
$$

Combining (14) and (15) yields that (10) is satisfied, which concludes the proof of the first statement of Theorem 1.

Second, consider the case where Assumption 2 is satisfied. Then, there exists an infinite sequence of time instants $\{t_i\} \subseteq \mathbb{I}_{\geq 0}$ such that $\beta(t_i) \leq \beta_\infty$ for all $t_i$ and some $\beta_\infty < \infty$. Using the fact that $\sum_{t=0}^{t-1} \gamma(t) = \beta(t) - \beta_0$ by definition of $\gamma$, and $\gamma(t) \leq c$ for all $t \in \mathbb{I}_{\geq 0}$ by assumption, (13) gives

$$
\limsup_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \{[(1 + c)e(t) - \ell(x(t), u(t))] \}
$$

Taking averages on both sides, we have

$$
\liminf_{T \to \infty} \frac{1}{T} \left( \int_0^T V(x(t), \beta(t)) dt - V(x_0, \beta_0) \right) \leq \liminf_{t \to \infty} \frac{1}{t_i} \left( V(x(t_i), \beta(t_i)) - V(x_0, \beta_0) \right) \leq \liminf_{t \to \infty} \frac{1}{t_i} \left( (t_i + \beta_\infty - \beta_0) \lambda_\infty + \sum_{t=0}^{t_i-1} [(1 + c)e(t) - \ell(x(t), u(t))] \right)
$$

where the fourth inequality is again due to the fact that $\lim_{t \to \infty} e(t) = 0$ and $e(t) \leq e(0) < \infty$ for all $t \in \mathbb{I}_{\geq 0}$, and the last inequality follows from the fact that $\{t_i\}$ is an infinite subsequence of $\mathbb{I}_{\geq 0}$. Combining (16) with (15) (which is valid independent of whether Assumption 1 or 2 is satisfied) results in (11), which concludes the proof of the second statement of Theorem 1.

Remark 2: Assumption 2 is weaker than requiring $\beta$ to be bounded. In fact, if Assumption 2 is strengthened to $\beta$ being bounded, then again the stronger conclusion (10) instead of (11) follows. Namely, if $\beta(t) \leq \beta_\infty$ for all $t \in \mathbb{I}_{\geq 0}$, then the calculations in (16) hold for all $t \in \mathbb{I}_{\geq 0}$ and not only for the subsequence $\{t_i\} \subseteq \mathbb{I}_{\geq 0}$. Then, the second-to-last line of (16) together with (15) imply that (10) is satisfied. □

IV. SELF-TUNING TERMINAL WEIGHTS

In this section, we propose several self-tuning update rules for the terminal weight $\beta$. Namely, we assume that $\beta$ evolves according to the formula

$$
\beta(t + 1) = B(\beta(t), x(t), \lambda(t)), \quad \beta(0) = \beta_0 \geq 0. \quad (17)
$$

Having established that $\lambda_\infty$ is an upper bound for the average performance of the closed-loop system, we want to find conditions such that bounds for $\lambda_\infty$ can be guaranteed. In this respect, we first formulate general conditions on update rules (Section IV-A), before presenting specific update rules satisfying these conditions (Section IV-B). Some of these update rules are actually more general than the formula (17) as they also include some memory of the past evolution of $\beta$.

A. Specifications for update rules $B$

Let $\omega_B(x_0)$ be the $\omega$-limit set of the closed-loop state sequence (9) starting at $x_0$ and using the update rule $B$ (17), i.e., $\omega_B(x_0) := \{ y \in \mathbb{Z}_x : \exists t_n \to +\infty \text{ s.t. } x(0) = x_0 \text{ and } \lim_{n \to \infty} x(t_n) = y \}$, where $x(\cdot)$ is the closed-loop solution arising from (9) and (17). We then have the following result:

**Theorem 2:** (i) Suppose that the update rule $B$ is such that for all sequences $x(\cdot)$ and $\lambda(\cdot)$, regarded as open-loop input signals in (17), it holds that

$$
\lambda_\infty - \inf_{t \to \infty} \ell_{\min}(x(t)) > 0 \Rightarrow \liminf_{t \to \infty} \ell_{\min}(x(t)) = \infty. \quad (18)
$$

Then, for the closed-loop system (9) and (17), it holds that

$$
\lambda_\infty = \lim_{t \to \infty} \ell_{\min}(x(t)) \leq \inf_{y \in \omega_B(x_0)} \ell_{\min}(y). \quad (19)
$$

(ii) Suppose that the update rule $B$ is such that for all sequences $x(\cdot)$ and $\lambda(\cdot)$, regarded as open-loop input signals in (17), it holds that

$$
\lambda_\infty - \sup_{t \to \infty} \ell_{\min}(x(t)) > 0 \Rightarrow \limsup_{t \to \infty} \ell_{\min}(x(t)) = \infty, \quad (20)
$$

Then, for the closed-loop system (9) and (17), it holds that

$$
\lambda_\infty = \lim_{t \to \infty} \sup_{y \in \omega_B(x_0)} \ell_{\min}(y). \quad (21)
$$

Remark 3: Conditions (18) and (20) in Theorem 2 have to be understood for “open-loop” sequences $x(\cdot)$, $\lambda(\cdot)$ and $\beta(\cdot)$, i.e., when some (given) sequences $x(\cdot)$ and $\lambda(\cdot)$ are regarded as open-loop input signals in the update rule B (17). The assertions (19) and (21), on the other hand, hold for “closed-loop” sequences $x(\cdot)$, $\lambda(\cdot)$ and $\beta(\cdot)$ resulting from the closed-loop system (9) and (17). □

Proof of Theorem 2: (i) First, we show that $\lambda_\infty - \inf_{t \to \infty} \ell_{\min}(x(t)) \leq 0$ holds for the closed-loop system (9) and (17) if (18) is satisfied. Namely, assume for contradiction that $\lambda_\infty - \inf_{t \to \infty} \ell_{\min}(x(t)) > 0$ holds for the
closed-loop system (9) and (17). We can then use the (closed-loop) sequences \( x(\cdot) \) and \( \lambda(\cdot) \) in (18) to conclude that the corresponding (closed-loop) terminal weight sequence \( \beta(\cdot) \) satisfies \( \lim_{t \to -\infty} \beta(t) = \infty \). However, in [15, Proposition 2] it was shown that for each \( \varepsilon > 0 \), if \( \beta > a/\varepsilon \) for some finite constant \( a > 0 \), then \( \ell(x^*(N|t), u^*(N|t)) \leq \ell_{\min}(x(t)) + \varepsilon \) for all \( x(t) \). But then, as \( \liminf_{t \to -\infty} \beta(t) = \infty \), by our assumption. Thus it holds that \( \lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) \) which contradicts our assumption. Thus it holds that \( \lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) \leq 0 \), which by the definition of \( \lambda_{\infty} \) and \( \ell_{\min} \) implies that \( \lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) \) exists and \( \lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) = 0 \). This establishes the equality in (19). Second, according to the \( t \to -\infty \) limit set \( \omega_{B}(x_0) \), for each \( y \in \omega_{B}(x_0) \) and each \( \varepsilon > 0 \) there exists an infinite sequence of time instants \( \{t^\alpha_n\} \) such that \( x(t^\alpha_n) \in B_{\varepsilon}(y) \cap \mathbb{Z}_G \). But this implies that

\[
\lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) = \lim_{t \to -\infty} \ell_{\min}(x(t^\alpha_n)) \leq \ell_{\min}(y, \varepsilon).
\]

As this holds for each \( \varepsilon > 0 \) and each \( y \in \omega_{B}(x_0) \), we obtain

\[
\lambda_{\infty} = \lim_{t \to -\infty} \ell_{\min}(x(t)) \leq \inf_{y \in \omega_{B}(x_0)} \ell_{\min}(y),
\]

which establishes statement (i) of the Theorem.

(ii) The proof of this statement is similar to the proof of statement (i). We first show that \( \lambda_{\infty} = \limsup_{t \to -\infty} \ell_{\min}(x(t)) \leq 0 \) for the closed-loop system (9) and (17) if (20) is satisfied. Namely, assume for contradiction that \( \lambda_{\infty} = \limsup_{t \to -\infty} \ell_{\min}(x(t)) > 0 \) for the closed-loop system (9) and (17). Then we can use the (closed-loop) sequences \( x(\cdot) \) and \( \lambda(\cdot) \) in (20) to conclude that the corresponding (closed-loop) terminal weight sequence \( \beta(\cdot) \) satisfies \( \limsup_{t \to -\infty} \beta(t) = \infty \). However, in [15, Proposition 2] it was shown that for each \( \varepsilon > 0 \), if \( \beta > a/\varepsilon \) for some finite constant \( a > 0 \), then \( \ell(x^*(N|t), u^*(N|t)) \leq \ell_{\min}(x(t)) + \varepsilon \) for all \( t \). But then, as \( \limsup_{t \to -\infty} \beta(t) = \infty \), we obtain that \( \lambda_{\infty} = \limsup_{t \to -\infty} \ell_{\min}(x(t)) \), which contradicts our assumption. By definition of \( \lambda_{\infty} \) and \( \ell_{\min} \), the above inequality has to hold with equality, i.e., \( \lambda_{\infty} = \limsup_{t \to -\infty} \ell_{\min}(x(t)) \), which establishes the equality in (21). Now let \( \{t_i\} \) be an infinite sequence of time instants such that \( \lim_{t_i \to -\infty} \ell_{\min}(x(t_i)) \) exists and satisfies \( \lim_{t \to -\infty} \ell_{\min}(x(t)) = \limsup_{t \to -\infty} \ell_{\min}(x(t)) \). According to the \( t \to -\infty \) limit set \( \omega_{B}(x_0) \), there exists a point \( y^* \in \omega_{B}(x_0) \) and for each \( \varepsilon > 0 \) an infinite subsequence \( \{t_r\} \) of the sequence \( \{t_i\} \) such that \( x(t_r) \in B_{\varepsilon}(y^*) \cap \mathbb{Z}_G \). But this implies that

\[
\limsup_{t \to -\infty} \ell_{\min}(x(t)) \leq \limsup_{t \to -\infty} \ell_{\min}(x(t_r)) \leq \ell_{\min}(y^*, \varepsilon).
\]

But as this holds for each \( \varepsilon > 0 \), we obtain

\[
\lambda_{\infty} = \limsup_{t \to -\infty} \ell_{\min}(x(t)) \leq \lim_{t \to -\infty} \ell_{\min}(x(t_r)) \leq \ell_{\min}(y^*),
\]

which establishes statement (ii) of the Theorem. \( \square \)

In Theorem 2, condition (20) is weaker than (18) and hence more update rules \( B \) will be likely to fulfill it. However, then also the resulting conclusion (21) which can be made is weaker than (19). Namely, in case (i), we can ensure that the best achievable steady-state cost along the closed-loop system converges, i.e., \( \lim_{t \to -\infty} \ell_{\min}(x(t)) \) exists. Furthermore, \( \lambda_{\infty} \) is equal to this limit, which is at least as good as the minimum of the best robustly achievable steady-state cost on the \( t \to -\infty \) limit set of the closed-loop system. On the other hand, in case (ii), we cannot necessarily ensure that \( \lim_{t \to -\infty} \ell_{\min}(x(t)) \) exists, but only that \( \lambda_{\infty} \) is equal to \( \limsup_{t \to -\infty} \ell_{\min}(x(t)) \), which in turn is at least as good as the supremum of the best robustly achievable steady-state cost on the \( t \to -\infty \) limit set of the closed-loop system. Furthermore, we remark that if \( \omega_{B}(x_0) \) is just a singleton, or, more general, if \( \ell_{\min}(y_1) = \ell_{\min}(y_2) \) for all \( y_1, y_2 \in \omega_{B}(x_0) \), then the right hand sides of (21) and (19), i.e., the upper bounds for \( \lambda_{\infty} \), are the same.

**B. Several update rules \( B \) and their properties**

In the following, we propose and discuss several different update rules for \( \beta \) which ensure that the conditions of Theorems 1 and 2 are satisfied. We start with rather simple update rules which lead to monotonically increasing \( \beta \). To this end, let

\[
\delta(t) := \ell(x^*(N|t), u^*(N|t)) - \ell_{\min}(x(t)).
\]  

- Update rule 1: \( B_1(\beta(t), x(t), \lambda(t)) := \beta(t) + d \) for some \( d > 0 \).
- Update rule 2: \( B_2(\beta(t), x(t), \lambda(t)) := \beta(t) + \alpha(\delta(t)) \) for some\(^5\) \( \alpha \in \mathcal{K} \).

The appeal of update rules 1 and 2 is clearly their simplicity. A further advantage of update rule 1 is the fact that in contrast to update rule 2, \( \ell_{\min}(x(t)) \) does not have to be known at each time \( t \). However, with update rule 1, in any case \( \beta \to \infty \), which might not be desirable.

**Lemma 1:** The update rules 1 and 2 are such that (18) holds; furthermore, for update rule 2, the sequence \( \beta(\cdot) \) resulting from the closed-loop system (9) and (17) satisfies Assumption 1.

**Proof:** With update rule 1, (18) is trivially satisfied as \( \liminf_{t \to -\infty} \beta(t) = \infty \) independent of the behavior of \( x(\cdot) \) and \( \lambda(\cdot) \). For update rule 2, consider the following. If \( \lambda_{\infty} = \liminf_{t \to -\infty} \ell_{\min}(x(t)) > 0 \) for some sequences \( x(\cdot) \) and \( \lambda(\cdot) \), then \( \delta(t_i) \geq c > 0 \) for an infinite sequence of time instants \( t_i \) and some constant \( c > 0 \). But this immediately implies that for the corresponding sequence \( \beta(\cdot) \), we have \( \liminf_{t \to -\infty} \beta(t) = \infty \) as \( \alpha \in \mathcal{K} \). Hence with both update rules 1 and 2, (18) is satisfied. In order to establish the second claim of the lemma, consider the following. For update rule 2, we have \( \gamma(t) = \beta(t+1) - \beta(t) = \alpha(\delta(t)) \). But \( \delta(t) \) and hence also \( \gamma(t) \) is bounded above by some finite constant due to continuity of \( \ell \) and compactness of \( \mathbb{Z}_G \), i.e., \( \gamma(t) \leq c < \infty \) for all \( t \in \mathbb{Z}_G \). Furthermore, \( \beta(t) \geq \beta_0 =: \beta \) for all \( t \geq 0 \). Finally, as (18) is satisfied, by Theorem 2(1)

\(^5\)A function \( \alpha: [0, \infty) \to [0, \infty) \) is of class \( \mathcal{K} \) if \( \alpha \) is continuous, strictly increasing, and \( \alpha(0) = 0 \).
we conclude that \( \lim_{t \to \infty} \delta(t) = 0 \) for the closed-loop system (9) and (17), and hence also \( \lim_{t \to \infty} \gamma(t) = 0 \) as \( \alpha \in \mathcal{K} \). Hence the sequence \( \gamma(t) \) resulting from the closed-loop system (9) and (17) satisfies Assumption 1.

We now turn our attention to slightly more complex update rules which are nonmonotonic, but allow for a reset of \( \beta \). Such nonmonotonic update rules are desirable as \( \beta \) might stay smaller, which, as mentioned above, might be good for performance reasons. Moreover, one might also have a greater robustness for \( \beta \) to stay bounded in case of disturbances (see, e.g., Example 2 in Section VI).

- **Update rule 3:** Let \( \alpha_1, \alpha_2 \in \mathcal{K} \).

  \[
  B_3(\beta(t), x(t), \lambda(t)) := \begin{cases}
  1 & \text{if } C_3(t) \leq 0, \\
  \beta(t) + \alpha_2(\delta(t)) & \text{else},
  \end{cases}
  \]

  where \( C_3(0) = 0 \) and for each \( t \in I_{\geq 1} \), \( C_3(t) := \ell(x^*(N(t)), u^*(N(t)) - \ell(x^*(N|t_{last}), u^*(N|t_{last})) + \alpha_1(\delta(t)) \) with \( t_{last} := \max_{s \leq t, \beta(s)} = 1 \).

  **Lemma 2:** Update rule 3 is such that (20) holds; moreover, the sequence \( \beta(t) \) resulting from the closed-loop system (9) and (17) satisfies at least one of Assumptions 1 and 2.

  **Proof:** We start by proving the first claim of the lemma. If \( \lambda_\infty = \limsup_{t \to \infty} \ell_{\min}(x(t)) > 0 \) for some sequences \( x(t) \) and \( \lambda(t) \), then there exists \( \ell \) such that \( \delta(t) \geq \epsilon > 0 \) for all \( t \geq \ell \) and some \( \epsilon > 0 \). We will show that in this case the number of resets to 1 of the corresponding sequence \( \lambda(t) \) is finite. Namely, assume it was not. Then, both \( \ell(x^*(N|t), u^*(N|t)) \) as well as \( \ell(x^*(N|t_{last}), u^*(N|t_{last})) \) converge to \( \lambda_\infty \). But this implies that there exists \( \ell \geq \ell \) such that \( C_3(t) > 0 \) for all \( t \geq \ell \) (as \( \delta(t) \geq \epsilon > 0 \)), i.e., the reset condition is not satisfied anymore, which gives a contradiction. Hence the number of resets of \( \lambda(t) \) is finite. But then according to the definition of \( B_3 \), we immediately obtain that \( \liminf_{t \to \infty} \beta(t) = \limsup_{t \to \infty} \beta(t) = \infty \), i.e., (20) is satisfied. To prove the second claim of the lemma, consider the following. First, the definition of \( B_3 \) implies that \( \beta(t) \geq \min \{ \beta_0, 1 \} \geq 0 \). Second, whenever \( \beta(t) = 1 \), we have \( \gamma(t) = 0 \), which is bounded above by some finite constant as established in the proof of Lemma 1. Hence \( \gamma(t) \leq \epsilon < \infty \) for all \( t \in I_{\geq 0} \).

- **Update rule 4:** Let \( \beta(t), x(t), \lambda(t) := \beta(t) + \alpha_2(\delta(t)) \).

  **Lemma 3:** Update rules 4 and 5 are such that (20) holds. Moreover, for update rule 4, the sequence \( \beta(t) \) resulting from the closed-loop system (9) and (17) satisfies Assumption 1, whereas it satisfies at least one of Assumptions 1 and 2 for update rule 5.

  **Proof:** The proof is similar to the proof of Lemma 2. We start proving the first claim. If \( \lambda_\infty = \limsup_{t \to \infty} \ell_{\min}(x(t)) > 0 \) for some sequences \( x(t) \) and \( \lambda(t) \), then, as \( \limsup_{t \to \infty} \nu(t) = 1 \), there exists \( \ell > 0 \) such that \( \delta(t) > \epsilon > 0 \) for all \( t \geq \ell \) and some \( \epsilon > 0 \). For update rule 4, this immediately implies that \( \liminf_{t \to \infty} \beta(t) = \limsup_{t \to \infty} \beta(t) = \infty \). For update rule 5, one can show with the same reasoning as in the proof of Lemma 2 that the number of resets of \( \beta \) in this case is finite, which again yields \( \liminf_{t \to \infty} \beta(t) = \limsup_{t \to \infty} \beta(t) = \infty \). Hence with both update rules 4 and 5, (20) is satisfied. The second statement of the lemma can be proven analogous to Lemma 2. Namely, for the sequence \( \beta(t) \) resulting from the closed-loop system (9) and (17), one can show by...
contradiction that \( \limsup_{t \to -\infty} \gamma(t) \leq 0 \) for update rule 4, and for update rule 5 that either \( \limsup_{t \to -\infty} \gamma(t) \leq 0 \) or the number of resets is infinite, which implies that \( \liminf_{t \to -\infty} \beta(t) = 1 < \infty \). Finally, the constants \( c \) and \( \beta \) can be calculated as in the proof of Lemmas 1 and 2, respectively.

Next, we would like to find an update rule which is non-monotonic but still satisfies the stronger condition of statement (i) of Theorem 2, i.e., (18). To this end, consider the following update rule, which is a slight modification of update rule 3.

- **Update rule 6:** Let \( \alpha_1, \alpha_2 \in \mathcal{K} \).

\[
B_6(\beta(t), x(t), \lambda(t)) := \begin{cases} 
1 & \text{if } C_6(t) \leq 0, \\
\beta(t) + \alpha_2(\delta(t)) & \text{else,}
\end{cases}
\]

where \( C_6(t) := \ell(x^*(N|t), u^*(N|t)) - \ell(x^*(N|t_{\text{last}}), u^*(N|t_{\text{last}})) + \alpha_1(\max_{t_{\text{last}} + 1 \leq s \leq t} \delta(s)) \) and \( t_{\text{last}} := \max_{s \leq t \beta(s)} s = 1 \).

**Lemma 4:** Update rule 6 is such that (18) holds; moreover, the sequence \( \beta(\cdot) \) resulting from the closed-loop system (9) and (17) satisfies Assumption 1.

**Proof:** We again start by proving the first claim. If \( \lambda_\infty = \liminf_{t \to -\infty} \ell_{\text{min}}(x(t)) > 0 \) for some sequences \( x(\cdot) \) and \( \lambda(\cdot) \), then \( \delta(t) \geq c > 0 \) for an infinite sequence of time instants \( t_i \) and some constant \( c > 0 \). Again, we show that in this case the number of resets of the corresponding sequence \( \beta(\cdot) \) is finite. Namely, assume it was not. Then, both \( \ell(x^*(N|t), u^*(N|t)) \) as well as \( \ell(x^*(N|t_{\text{last}}), u^*(N|t_{\text{last}})) \) converge to \( \lambda_\infty \), i.e., there exists \( \ell > 0 \) such that \( \ell(x^*(N|t), u^*(N|t)) - \ell(x^*(N|t_{\text{last}}), u^*(N|t_{\text{last}})) \geq -\alpha_1(\ell/2) \) for all \( t \geq \ell \). Now consider the next time instant \( t' \geq \ell \) such that \( \delta(t') \geq c \). Then, for all \( t \geq t' \), the reset condition for \( \beta(\cdot) \) will not be satisfied anymore as (by induction) we obtain

\[
C_6(t) \geq -\alpha_1(\ell/2) + \alpha_1(\max_{t_{\text{last}} + 1 \leq s \leq t} \delta(s)) \geq -\alpha_1(\ell/2) + \alpha_1(c) > 0,
\]

where the last inequality follows as \( \alpha_1 \in \mathcal{K} \). Hence the number of resets of \( \beta(\cdot) \) is finite. But then the definition of \( B_6 \) yields that \( \liminf_{t \to -\infty} \beta(t) = \infty \), i.e., (18) is satisfied. To show that Assumption 1 is satisfied, one can calculate the constants \( \beta \) and \( c \) as in the proof of Lemma 2; furthermore, as (18) holds, by Theorem 2(i) we conclude that \( \liminf_{t \to -\infty} \delta(t) = 0 \) for the closed-loop system (9) and (17), and hence \( \limsup_{t \to -\infty} \gamma(t) \leq 0 \) due to the definition of \( B_6 \).

**Remark 5:** In update rules 2-6, the \( K \)-functions \( \alpha \) (respectively, \( \alpha_1 \) and \( \alpha_2 \)) are tuning parameters determining how fast \( \beta \) grows and how often it is reset. If these \( K \)-functions are "flat enough", \( \beta \) will grow more slowly and will be reset more often, respectively. We will further illustrate this property with some examples in Section VI.

**Remark 6:** In update rules 3, 5 and 6, one could also reset \( \beta \) to some value \( \beta \geq 0 \) different than 1. Resetting \( \beta \) to 1 is a canonical choice, as with \( \beta = 1 \) the terminal state/input pair is weighted equally as all other state/input pairs in the cost function (7).

**V. Discussion**

A few comments on the obtained results as well as the benefits and limitations of the proposed approach are in order. First, Theorem 1 established that \( \lambda_\infty \) is an upper bound for the closed-loop average cost for possibly time-varying terminal weights \( \beta \). In general, \( \lambda_\infty \) depends both on the initial values \( \lambda(0) \) and \( x(0) \) as well as on the update rule \( B \) for the terminal weight \( \beta \). By definition of \( \ell(x, u) \) in (6) and the fact that \( \ell(\cdot) \) is nonincreasing, immediate a priori bounds for \( \lambda_\infty \) are \( \ell(x, u) \leq \lambda_\infty \leq \lambda(0) \). Hence it would be tempting to choose \( \lambda(0) \) as small as possible. However, a small value of \( \lambda(0) \) might result in a degradation of the (transient) performance, and moreover, the feasible region critically depends on \( \lambda(0) \); if we choose \( \lambda(0) = \ell(x, u) \), then (8e) reduces to \( x(N|t) = x_s \), and hence the feasible region is that of a fixed terminal point setting (and is empty if \( \lambda(0) < \ell(x, u) \)). On the other hand, the largest feasible region is obtained if \( \lambda(0) \) is chosen such that \( \lambda(0) \geq \max_{x,u} c(x,u) = \ell(x, u) \), as then each steady-state in \( \mathbb{Z} \) can serve as a terminal state at time 0. Hence a good tradeoff for \( \lambda(0) \) has to be found.

Theorem 2 shows that \( \lambda_\infty \) can be upper bounded in terms of the best robustly achievable steady-state cost of points in the \( \omega \)-limit set of the closed-loop system (which in general depends on the specific update rule used). Each of the presented update rules has different advantages concerning simplicity of implementation and strength of the results which can be obtained. According to Theorems 1 and 2, the strongest results can be proven for update rules 2 and 6, as both Assumption 1 and (18) are satisfied. On the other hand, the weakest results\(^6\) are obtained for update rules 3 and 5, as only Assumption 2 and (20) are satisfied. Regarding implementation issues, update rules 2 and 4 would be beneficial for their simplicity, as no reset condition has to be checked. On the other hand, as already discussed, resetting \( \beta \) is beneficial in order not to let the terminal weight grow unnecessarily big, which might degrade performance. Furthermore, as already mentioned above, a big advantage of update rules 4 and 5 is that \( \ell_{\text{min}}(x(t)) \) does not have to be calculated at each time step \( t \), but only on an infinite subsequence \( \{t_i\} \). Note that in the special case where \( \ell_{\text{min}}(x) = \ell(x, u,s) \) for all \( x \in \mathbb{Z} \) (such as in Example 2 in Section VI), update rules 2 and 4 as well as 3 and 5 coincide, as \( \delta(t) = \delta(t) \) in this case, and \( \ell_{\text{min}}(x) \) does not have to be calculated at all online. Finally, we remark that while update rule 6 yields, as just discussed, the strongest results of those update rules exhibiting resets, its reset condition is also the strictest one, and hence in general resets do not occur as frequently compared to update rules 3 and 5.

\(^6\)For update rule 1, which clearly is the most simple update rule and which is rather stated for motivation of the subsequent ones, Theorem 1 cannot be evoked at all, as neither Assumptions 1 or Assumptions 2 are satisfied.
summary, when choosing a specific update rule $B$ for the terminal weight $\beta$, several different aspects have to be traded off against each other such as to decide which one is best suited for the specific problem considered.

The results presented in this paper are valid for general (nonlinear) system dynamics and cost functions. An interesting question, which is, however, beyond the scope of this paper, would be to examine whether for specific system classes and cost functions (e.g., linear systems with convex cost functions), sharper results on $\lambda_\infty$ can be obtained. In particular, it would be interesting to determine under what conditions on the system dynamics, the cost function, and the prediction horizon it follows that $\lambda_\infty = \ell(x_s, u_s)$ (at least approximately). As mentioned in the Introduction, a first result in this direction was obtained in [15, Algorithm 3] for large constant terminal weights under an additional controllability/reachability assumption and by further modifying the MPC algorithm, i.e., if necessary, following the previously optimal solution.

VI. EXAMPLES

Example 2: Consider the system $x(t+1) = (1 - u(t))x(t)$ with state and input constraint set $\mathcal{Z} = \mathbb{X} \times \mathbb{U} = [0, 1]^2$, cost function $\ell(x, u) = x + (1/2)u^2$ and prediction horizon $N = 1$. The optimal steady-state is given by $(x_s, u_s) = (0, 0)$, and for all $x \in \mathbb{X}$, we obtain $\ell_{\min}(x) = \ell(x_s, u_s) = 0$. It is straightforward to calculate that the optimal solution to problem (7)–(8) with $N = 1$ is given by $u^*(0|t) = \min(\beta(t)x(t), 1)$, $u^*(1|t) = 0$. The resulting closed-loop system is then given by

$$x(t+1) = (1 - \min(\beta(t)x(t), 1))x(t),$$

with $\beta(t)$ according to the specific used update rule. For each of the above discussed update rules and each initial condition $x_0 \in \mathbb{X}$ and $\beta_0 \geq 0$, the $\omega$-limit set is given by $\omega_{B}(x_0) = \{0\}$, and $\max_{y \in \omega_{B}(x_0)}\ell_{\min}(y) = \min_{y \in \omega_{B}(x_0)}\ell_{\min}(y) = 0$. Hence for each of the above update rules, by Theorem 2 we conclude that $\lambda_\infty = 0$. However, the closed-loop evolution of $\beta$ is quite different when using different update rules (see Figure 2(a)). As noted in Section V, update rules 2 and 4 as well as 3 and 5 coincide, as $\ell_{\min}(x) = \ell(x_s, u_s)$ for all $x \in \mathbb{X}$. For update rule 2 (respectively 4), one can show that when using $\alpha(r) = r$, $\beta$ also does not stay bounded, as $x$ converges to zero very slowly (blue curve in Figure 2(a)). On the other hand, when using $\alpha(r) = r^2$, one observes that $\beta$ stays bounded and converges (magenta curve in Figure 2(a)).

When using update rule 3 (respectively 5) with $\alpha_1(r) = r^2$ and $\alpha_2(r) = r$, one can show that $\beta \equiv 1$, as the reset condition is always fulfilled (green curve in Figure 2(a)). For $\alpha_1(r) = \alpha_2(r) = r$, on the other hand, one observes kind of a sawtooth behavior of $\beta$ (red curve in Figure 2(a)).

As discussed earlier, an advantage of nonmonotonic update rules is that they might be more robust to disturbances. Namely, consider again the same update rules as used in Figure 2(a), but now an additive random disturbance (uniformly distributed over the interval $[0, 0.2]$) acts on the system (24), as one can see from the simulations (see Figure 2(b)), the monotonic update rule 2 (respectively 4) now leads to an unbounded $\beta$, for both the choices $\alpha(r) = r$ and $\alpha(r) = r^2$. On the other hand, update rule 3 (respectively 5) leads to a bounded $\beta$.

Example 3: Consider the system $x(t+1) = x(t)/(1 + x(t)u(t))$ with state and input constraint set $\mathcal{Z} = \mathbb{X} \times \mathbb{U} = [0, 1]^2$, cost function $\ell(x, u) = x^2 + 2u + xu^2$ and prediction horizon $N = 1$. For all $x \in \mathbb{X}$, the best achievable steady-state is $(x, u) = (x/(x + 1), 0)$, i.e., $\ell_{\min}(x) = x^2/(x + 1)^2$ for all $x \in \mathbb{X}$, and the optimal steady-state is given by $(x_s, u_s) = (0, 0)$. It is straightforward to calculate that the optimal solution to problem (7)–(8) with $N = 1$ is given by

$$u^*(0|t) = \min\left\{1, \max \left\{\frac{\sqrt{\beta(t)x(t)^3}}{x(t)} - 1}{0}\right\}\right.,$$

$$u^*(1|t) = 0,$$

for all $x(t) \neq 0$. The resulting closed-loop system is then
given by
\[ x(t + 1) = \frac{x(t)}{1 + x(t) \min \left\{ 1, \max \left\{ \sqrt[n]{\beta(t)x(t)^{\ell - 1}}, 0 \right\} \right\}}, \]
with \( \beta(t) \) according to the specific used update rule. From (25), it follows that if \( \beta \) is bounded, i.e., \( \beta(t) \leq \beta \) for some \( \beta < \infty \) and all \( t \in \mathbb{N} \) (which in particular would be the case if a constant terminal weight was used), then \( x \) remains constant once \( x \leq 1/\sqrt[n]{\beta} \). So in order to converge to the optimal steady-state and to obtain \( \lambda_{\infty} = 0 \), we necessarily need that \( \lim_{t \to \infty} \sup_{t \to \infty} \beta(t) = \infty \). This happens for all six update rules proposed in Section IV-B, as according to Lemmas 1–4, for each of these update rules either (18) or (20) is satisfied. Figure 3 shows closed-loop evolutions of \( \beta \) for different update rules.

**Example 4:** This example is a simple illustration of the fact why in Theorem 2, we only can ensure that \( \lambda_{\infty} \) converges to a value not greater than the minimum (respectively, maximum) of the best robustly achievable steady-state cost on the \( \omega \)-limit set. Namely, consider the system
\[
\begin{align*}
x_1(t + 1) &= x_1(t) + u(t), \\
x_2(t + 1) &= x_2(t) - |\sin(x_2(t))| + u(t).
\end{align*}
\]

The equilibria of this system are given by \( u = 0 \), \( x_2 = k\pi \) with \( k \in \mathbb{N} \) and \( x_1 \) arbitrary. Furthermore, we consider state and input constraints \( Z = \mathbb{X} \times \mathbb{U} \), where \( \mathbb{X} = [-\pi, \pi] \) and \( \mathbb{U} = [-u_{\max}, u_{\max}] \), with \( u_{\max} \) such that \( 2u_{\max} + |\sin(u_{\max})| = \pi \), and a prediction horizon of \( N = 2 \). The cost function is \( \ell(x_2, u) = |u| + g(x_2) \) with \( g(x_2) \) as sketched in Figure 4. For each \( x_2 \in (0, \pi] \), the best reachable steady-state in two steps is \( x_2 = 0 \), and for each \( x_2 \in [-\pi, 0] \), the best reachable steady-state in two steps is \( x_2 = -\pi \). Thus, for each \( \varepsilon > 0 \) small enough, \( \ell_{\min}(0, \varepsilon) = \ell(0, 0) = g(0) = 1 \) and hence also \( \overline{\ell}_{\min}(0) = 1 \). On the other hand, \( \ell_{\min}(0) = \ell(-\pi, 0) = g(-\pi) = 0 < \overline{\ell}_{\min}(0) \), which means that \( \ell_{\min}(0, \varepsilon) \) is not continuous in \( \varepsilon \) at \( \varepsilon = 0 \).

**VII. CONCLUSIONS**

In this paper, we considered an economic MPC algorithm with self-tuning terminal cost. We proposed several update rules for the terminal weight \( \beta \) which resulted in an average performance which is at least as good as the best robustly achievable steady-state cost of the \( \omega \)-limit set of the corresponding closed-loop system, and we discussed the different properties, benefits and limitations of these update rules. In conclusion, we believe that using a self-tuning terminal cost can be a useful tool in the context of economic MPC. As already discussed in Section V, future research could include analyzing under what conditions on the system dynamics, the cost function and the prediction horizon possibly sharper (a priori) bounds on the closed-loop average performance can be obtained.

**REFERENCES**


