Performance analysis of economic MPC with self-tuning terminal cost

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Abstract—In this paper, we further analyze an economic model predictive control (MPC) algorithm with self-tuning terminal cost, which was recently proposed in [1]. It is shown that if a generalized terminal region constraint instead of a generalized terminal equality constraint is used, improved closed-loop asymptotic average performance bounds can be obtained. In particular, in contrast to the results in [1], these bounds can be obtained a priori. We discuss how the necessary parameters for the generalized terminal region setting can be calculated, and we illustrate our findings with a numerical example.

I. INTRODUCTION

Economic MPC is a variant of model predictive control which has received significant attention recently. In contrast to standard tracking MPC, where the control objective is to stabilize a given setpoint (or trajectory to be tracked), in economic MPC a general performance criterion (possibly representing the economics of a process) is used, which is not necessarily positive definite with respect to some setpoint. In [2–7], using different assumptions and/or additional (terminal) constraints, various properties of economic MPC were studied, such as average performance and convergence of the resulting closed-loop system, or fulfillment of average constraints. Also, successful implementations of economic MPC in various application contexts were reported recently, see, e.g., [8–10].

In order to overcome some of the limitations when using a (fixed) terminal point or terminal region constraint, an MPC framework using a generalized terminal state constraint has been proposed, first in the context of tracking MPC [11, 12], and recently also in economic MPC [1, 13, 14]. This means that the endpoint of the predicted state sequence has to be equal to some arbitrary steady-state and not to a specific one. The main benefits of such a generalized terminal constraint setting are a possibly much larger region of attraction and a equal to some arbitrary steady-state and not to a specific one.

Our main result (see Theorem 3 in Section III) shows that if a generalized terminal region constraint instead of a generalized terminal equality constraint is used; this idea (defined, however, in a different way) has also been used in the context of tracking MPC (see, e.g., [11, 12]). Our main result (see Theorem 3 in Section III) shows that in this case, the closed-loop average performance is at least as good as the best robustly achievable steady-state cost of the -limit set of the resulting closed loop system (see Section II-C for further details), which can, in general, not be determined a priori.

In this paper, we overcome this limitation and show that improved closed-loop average performance guarantees can be given if a generalized terminal region constraint instead of a generalized terminal equality constraint is used; this idea (defined, however, in a different way) has also been used in the context of tracking MPC (see, e.g., [11, 12]).

Our main result (see Theorem 3 in Section III) shows that in this case, the closed-loop average performance is at least as good as a value corresponding to a local minimum of the stage cost function restricted to the set of feasible steady-states. For linear systems with convex cost and constraints, this results in the average performance being at least as good as the optimal steady-state, which recovers results obtained for a fixed terminal constraint [2]. In Section IV, we discuss how the necessary parameters for the generalized terminal region setting can be calculated. A simple example illustrates our findings in Section V, before we give some concluding remarks in Section VI.

A. Notation

Let \( \mathbb{I}_{\geq 0} \) denote the set of nonnegative integers, and \( \mathbb{I}_{[a,b]} \) the set of all integers in the interval \([a,b] \subseteq \mathbb{R} \). We define \( B_\varepsilon(y) \) to be the ball of radius \( \varepsilon > 0 \) around the point \( y \in \mathbb{R}^n \), i.e., \( B_\varepsilon(y) := \{ x \in \mathbb{R}^n : |x - y| \leq \varepsilon \} \). For a function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \), \( g_x(y) \) denotes the gradient and \( g_{xx}(y) \) the Hessian of \( g \) with respect to \( x \), evaluated at the point \( y \in \mathbb{R}^n \). Given two sets \( A, B \subseteq \mathbb{R}^n \), the Minkowski set addition and Pontryagin set difference are defined as \( A + B := \{ a + b : a \in A, b \in B \} \) and \( A \ominus B := \{ a \in A : a + b \in \mathbb{R}^n \} \).
A \forall b \in B \}, \) respectively. For a symmetric matrix \( A \in \mathbb{R}^{n \times n} \), denote by \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) its minimum and maximum eigenvalue, respectively.

II. Economic MPC with self-tuning terminal cost

We consider discrete-time nonlinear systems of the form

\[ x(t + 1) = f(x(t), u(t)), \quad x(0) = x_0, \quad (1) \]

with \( x(t) \in X \subseteq \mathbb{R}^n \) and \( u(t) \in U \subseteq \mathbb{R}^m \) for all \( t \in I_{x_0} \), and \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is assumed to be continuous. The system is subject to (possibly coupled) state and input constraints \( (x(t), u(t)) \in Z \) for some compact set \( Z \subseteq X \times U \). Denote by \( Z_X \) the projection of \( Z \) on \( X \).

A. Generalized terminal state constraint

In [1], we proposed the following economic MPC algorithm with a self-tuning terminal weight, which is built on one introduced in [14] with fixed terminal weight and which we briefly recall here for the sake of completeness. At each time \( t \) with \( x := x(t) \), the following optimization problem is solved:

\[
\min_{u} J(x, u, \beta) = \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) + \beta(t)\ell(x(N|t), u(N|t)) \quad (2)
\]

subject to

\[
\begin{align*}
&x(0|t) = x \quad (3a) \\
&x(k+1|t) = f(x(k|t), u(k|t)) \quad k \in I_{[0,N-1]} \quad (3b) \\
&(x(k|t), u(k|t)) \in Z, \quad k \in I_{[0,N]} \quad (3c) \\
&(x(N|t) = f(x(N|t), u(N|t)), \quad (3d) \\
&\ell(x(N|t), u(N|t)) \leq \lambda(t). \quad (3e)
\end{align*}
\]

for some possibly time-varying terminal weight \( \beta \) and \( \lambda \) specified later. The notation \( x(-|t) \) and \( u(-|t) \) denote predicted state and input values (predicted at time \( t \)), respectively. The stage cost \( \ell : X \times U \to \mathbb{R} \) is assumed to be continuous; as \( Z \) is compact, we can without loss of generality assume that \( \ell(x, u) \geq 0 \) for all \( (x, u) \in Z \). As already discussed in the introduction, the main advantages of using a generalized terminal state constraint (3d) instead of a fixed terminal point constraint lie in the fact that a possibly much larger region of attraction is obtained, and that the optimal steady-state, which is normally used as a fixed terminal point constraint [2], does not have to be known a priori.

B. Generalized terminal region constraint

For reasons to become apparent later on, in this paper we consider a relaxed form of the proposed MPC algorithm (2)–(3). Namely, instead of requiring the terminal predicted state to be equal to some steady-state as in (3d), we require the terminal predicted state to lie in a terminal region \( \mathcal{X}^f(\bar{x}) \) around some steady-state \( \bar{x} \). This leads to the following optimization problem to be solved at each time instant \( t \) with \( x := x(t) \):

\[
\begin{align*}
\min_{u, \bar{x}(t), \bar{u}(t)} J(x, u, \beta) &= \sum_{k=0}^{N-1} \ell(x(k|t), u(k|t)) \\
&+ V_f(x(N|t), \bar{x}(t)) + \beta(t)\ell(\bar{x}(t), \bar{u}(t)) \quad (4)
\end{align*}
\]

subject to

\[
\begin{align*}
x(0|t) &= x \quad (5a) \\
x(k+1|t) &= f(x(k|t), u(k|t)) \quad k \in I_{[0,N-1]} \quad (5b) \\
(x(k|t), u(k|t)) \in Z, \quad k \in I_{[0,N-1]} \\
x(N|t) &\in \mathcal{X}^f(\bar{x}(t)), \quad \bar{x}(t) = f(\bar{x}(t), \bar{u}(t)) \quad (5d) \\
\ell(\bar{x}(t), \bar{u}(t)) &\leq \lambda(t). \quad (5e)
\end{align*}
\]

Denote the optimal solution to problem (4)–(5) by \( \bar{x}(t), \bar{u}(t) \) and \( u^*(0|t) := [u^*(0|t)^T, \ldots, u^*(N-1|t)^T]^T \) and the corresponding state sequence by \( \mathbf{x}^t(0) := \left[ x^t(0)^T, \ldots, x^t(N|t)^T \right]^T \). As usual in MPC, the first part of the optimal input sequence, \( u^*(0|t) \), is applied to the system at time \( t \). The optimal value function is denoted by \( V(x, \beta) := J(x, u^*, \beta) \). The parameter \( \lambda \) is updated according to the cost of the previous optimal steady-state around which the terminal region was built, i.e., the following closed-loop system is obtained:

\[
\begin{align*}
x(t + 1) &= f(x(t), u^*(0|t)), \\
\lambda(t + 1) &= \ell(\bar{x}^t(t), \bar{u}^t(t)) \quad (6)
\end{align*}
\]

Finally, the terminal weight \( \beta \) follows some general update rule \( B \):

\[
\beta(t + 1) = B(\beta(t), x(t), \lambda(t)). \quad (7)
\]

The interested reader is referred to [1] for different specific update rules \( B \) and a detailed discussion regarding initialization (of \( \lambda \) and \( \beta \)) and implementation issues.

Remark 1: As pointed out before, the constraints (5d)–(5f) are a generalization of (3d)–(3e). In fact, if a terminal equality constraint is used, i.e., \( \mathcal{X}^f(\bar{x}) := \{ \bar{x} \} \), then (5d) and (5f) reduce to (3d)–(3e); furthermore, in this case we can choose \( \bar{Z} = Z \), and hence (5e) reduces to the last constraint of (3c), i.e., for \( k = N \) (see below the definition of the set \( Z \) and a further discussion of the constraint (5e)). Also, without loss of generality the terminal cost \( V_f \) satisfies \( V_f(\bar{x}, \bar{x}) = 0 \), and hence the cost function (4) reduces to (2) in case that \( \mathcal{X}^f(\bar{x}) := \{ \bar{x} \} \).

We make the following assumption on the set \( Z \), the terminal region \( \mathcal{X}^f \) and the terminal cost \( V_f \).

Assumption 1: Let \( \alpha > 0 \) and \( P, Q > 0 \). For each steady-state \( (\bar{x}, \bar{u}) \in \bar{Z} \), there exists a terminal region of the form \( \mathcal{X}^f(\bar{x}) := \{ x \in \mathbb{R}^n : E(x, \bar{x}) \leq \alpha \} \) with \( E(x, \bar{x}) := (x - \bar{x})^TP(x - \bar{x}) \), a continuous auxiliary terminal control law

1For simplicity, we assume that \( u^*(t), \bar{x}^t(t), \bar{u}^t(t) \) are unique. If this is not the case, just assign a unique constant selection map to select one of the multiple minima.
\[ \kappa_f(x, \bar{x}) \text{ with } \kappa_f(x, \bar{x}) = \bar{u}, \text{ and a continuous terminal cost function } V_f(x, \bar{x}) \text{ such that the following is satisfied for all } x \in \mathbb{R}^f(\bar{x}) : \]

(i) \((x, \kappa_f(x, \bar{x})) \in \mathbb{Z}_r\),

(ii) \(E(f(x, \kappa_f(x, \bar{x})), \bar{x}) - E(x, \bar{x}) \leq -(x - \bar{x})^T Q(x - \bar{x})\),

(iii) \(V_f(f(x, \kappa_f(x, \bar{x})), \bar{x}) - V_f(x, \bar{x}) \leq -\ell(x, \kappa_f(x, \bar{x})) + \ell(\bar{x}, \bar{u})\).

Remark 2: For a fixed \(\bar{x}\), conditions (i)–(iii) of Assumption 1 reduce to standard conditions imposed when using a terminal cost/region framework, both in the case of tracking and economic MPC [3, 15]. In (ii), we require something slightly stronger than invariance of the terminal region, namely that the terminal region is contractive if the local controller is applied; this is crucial for our main results later on. Note that it is sufficient if this holds for some arbitrary positive definite \(Q\), i.e., some \(Q\) with eigenvalues whose real part is arbitrarily close to 0. We will discuss in Section IV how Assumption 1 can be satisfied for all \((\bar{x}, \bar{u}) \in \mathbb{Z}_r\).

The set \(\mathbb{Z} \subseteq \mathbb{Z}\) in (5e) has to be defined such that condition (i) in Assumption 1 can be satisfied for all steady-states \((\bar{x}, \bar{u}) \in \mathbb{Z}_r\), i.e., such that state and input constraints are satisfied in the terminal region around each steady-state \((\bar{x}, \bar{u}) \in \mathbb{Z}_r\). In general, \(\mathbb{Z}\) depends on the size of the terminal region, i.e., on \(\alpha\) (for given \(P\)). Namely, the larger \(\alpha\), the smaller \(\mathbb{Z}\) has to be. In Section IV, we show how both the terminal ingredients as well as \(\mathbb{Z}\) can be defined such that Assumption 1 is satisfied. There, we also further discuss the constraint (5e) and how the feasible sets of problems (2)–(3) and (4)–(5) are related. For clarity of presentation of the main ideas, we suppose in the following that Assumption 1 is satisfied (as stated) with a fixed \(\alpha\), i.e., for a constant size of the terminal regions. In Section IV, we also discuss how this can be relaxed (see Remark 4).

C. Conceptual performance bounds

In this section, we briefly review the results obtained in [1] and show that they carry over to the modified setting with a generalized terminal region. To this end, define the set of steady-states \((\bar{x}, \bar{u})\) such that a terminal region (as specified in Assumption 1) around \(\bar{x}\) is reachable in \(N > 0\) steps from a point \(y \in \mathbb{Z}_r\) as

\[ \mathcal{X}_N(y) := \{(\bar{x}, \bar{u}) \in \mathbb{Z}_r : \exists \bar{u} \in \mathbb{R}^N \text{ s.t. } x(0) = y, x(j + 1) = f(x(j), u(j)) \forall j \in [0, N-1], x(N) \in \mathbb{R}^f(\bar{x}), (x(j), u(j)) \in \mathbb{Z} \forall j \in [0, N-1], \bar{x} = f(\bar{x}, \bar{u})\}. \]

Note that for each \(y \in \mathbb{Z}_r\), the set \(\mathcal{X}_N(y)\) is compact as \(\mathbb{Z}\), \(\mathbb{Z}_r\) and the terminal regions \(\mathbb{R}^f(\bar{x})\) are compact and \(f\) is continuous. Denote the best achievable steady-state cost from a point \(y \in \mathbb{Z}_r\) by\(^2\)

\[ \ell_{\min}(y) := \min_{x,u} \ell(x, u) \text{ s.t. } (x, u) \in \mathcal{X}_N(y) \]

Furthermore, the best robustly achievable steady-state cost from a point \(y \in \mathbb{Z}_r\) is defined as follows. For each \(\varepsilon \geq 0\), denote by

\[ \ell_{\min}(y, \varepsilon) := \sup_{z \in B_{\varepsilon}(y) \cap \mathbb{Z}_r} \ell_{\min}(z) \]

the supremum of the best achievable steady-state cost on the set \(B_{\varepsilon}(y) \cap \mathbb{Z}_r\). With this, we define the best robustly achievable steady-state cost from a point \(y \in \mathbb{Z}_r\) as

\[ \ell_{\min}(y) := \lim_{\varepsilon \to 0} \ell_{\min}(y, \varepsilon). \]

Note that the limit in (11) exists as \(\ell_{\min}(y, \varepsilon)\) is monotonically nonincreasing when \(\varepsilon\) decreases to zero. From the definitions in (9) and (11), it immediately follows that for each \(y \in \mathbb{Z}_r\) we have \(\ell_{\min}(y) \leq \ell_{\min}(y); \) however, equality does not hold in general as \(\ell_{\min}(y, \varepsilon)\) is not necessarily continuous in \(\varepsilon\) at \(\varepsilon = 0\) (see [1] for a simple example of this fact).

Given the above, we can now recap the average performance results from [1] for the closed-loop system (6). To this end, note that from (5f) and (6), it follows that the sequence \(\lambda(t)\) is nonincreasing; as it is also bounded from below (as \(\ell(x, u) \geq 0 \text{ for all } (x, u) \in \mathbb{Z}_r\), it converges. Denote the limit by \(\lambda_\infty := \lim_{t \to \infty} \lambda(t) \geq 0\). Furthermore, we make the following assumption on the closed-loop terminal weight sequence \(\beta(\cdot)\) resulting from (6)–(7). To this end, let \(\gamma(t) := \beta(t + 1) - \beta(t)\).

Assumption 2: The sequence \(\beta(\cdot)\) satisfies \(\gamma(t) \leq c \text{ and } \beta(t) \geq \beta(t) \geq \beta\) for all \(t \in \mathbb{Z}_r\) and some constants \(c, \beta \in \mathbb{R}\), and \(\lim_{t \to \infty} \beta(t) < \infty\).

Assumption 3: The sequence \(\beta(\cdot)\) satisfies \(\gamma(t) \leq c \text{ and } \beta(t) \geq \beta\) for all \(t \in \mathbb{Z}_r\) and some constants \(c, \beta \in \mathbb{R}\), and \(\lim_{t \to \infty} \beta(t) < \infty\).

Theorem 1 ([1, Theorem 1]): Suppose that Assumption 1 is satisfied and the optimization problem (4)–(5) is feasible at \(t = 0\). Then it is feasible for all \(t \in \mathbb{Z}_r\). Consider the closed-loop system (6). If \(\beta(\cdot)\) satisfies Assumption 2, then

\[ \lim_{T \to \infty} \sup_{T} \sum_{t=0}^{T-1} \ell(x(t), u(t)) / T \leq \lambda_\infty. \]

If \(\beta(\cdot)\) satisfies Assumption 3, then

\[ \lim_{T \to \infty} \inf_{T} \sum_{t=0}^{T-1} \ell(x(t), u(t)) / T \leq \lambda_\infty. \]

Proof: As usual in MPC, a feasible solution to problem (4)–(5) at time \(t + 1\) is given by the end piece of the previously optimal solution appended by the terminal controller, i.e., \(\bar{u}(t + 1) := [\kappa_f(x^*(N)|t)|T, \ldots, \kappa_f(x^*(N)|t)|T, \bar{x}^*(N)|T, \bar{x}^*(T)|T, \ldots, \bar{x}^*(T)|T]\), and \((\bar{x}(t + 1), \bar{u}(t + 1)) := (\bar{x}^*(t), \bar{u}^*(t))\). This means that the optimization problem (4)–(5) is recursively feasible. With
this, we obtain
\[
V(x(t) + 1, \beta(t + 1)) - V(x(t), \beta(t)) \leq J(x(t) + 1, \bar{u}(t + 1), \beta(t + 1)) - J(x(t), \bar{u}(t), \beta(t)) = \ell(x(t), u(t)) + \ell(x^*(Nt), \kappa_f(x^*(Nt), \bar{x}^*(t))) + \lambda(x^*(Nt), \bar{x}^*(t)) \leq \ell(x(t), u(t)) + (1 + \gamma(t))\ell(\bar{x}^*(t), \bar{u}^*(t)). 
\] (14)

From here, the remainder of the proof follows the lines of the proof of Theorem 1 in [1].

Given that $\lambda_\infty$ is an upper bound on the closed-loop asymptotic average performance, it is of interest to determine the value of $\lambda_\infty$. In [1], the following rather conceptual results were obtained. Let $\omega_B(x_0)$ be the $\omega$-limit set of the closed-loop state sequence (6) starting at $x_0$, i.e., $\omega_B(x_0) := \{y \in \mathbb{R}^n : \exists t_n \to +\infty \text{ s.t. } x(t_n) = x_0 \text{ and } \lim_{n \to \infty} x(t_n) = y\}$, where $x(\cdot)$ is the closed-loop solution arising from (6).

**Theorem 2 ([1, Theorem 2]):** (i) Suppose that Assumption 1 is satisfied and the update rule $B$ is such that for all sequences $x(\cdot)$ and $\lambda(\cdot)$, regarded as open-loop input signals in (7), it holds that

$$\lambda_\infty - \lim_{t \to \infty} \ell_{\min}(x(t)) > 0 \Rightarrow \lim_{t \to \infty} \beta(t) = \infty.$$ (15)

Then, for the closed-loop system (6) and (7), it holds that $\lim_{t \to \infty} \ell_{\min}(x(t))$ exists and

$$\lambda_\infty = \lim_{t \to \infty} \ell_{\min}(x(t)) \leq \inf_{y \in \omega_B(x_0)} T_{\min}(y).$$ (16)

(ii) Suppose that Assumption 1 is satisfied and the update rule $B$ is such that for all sequences $x(\cdot)$ and $\lambda(\cdot)$, regarded as open-loop input signals in (7), it holds that

$$\lambda_\infty - \sup_{t \to \infty} \ell_{\min}(x(t)) > 0 \Rightarrow \lim_{t \to \infty} \beta(t) = \infty.$$ (17)

Then, for the closed-loop system (6) and (7), it holds that

$$\lambda_\infty = \sup_{t \to \infty} \ell_{\min}(x(t)) \leq \sup_{y \in \omega_B(x_0)} T_{\min}(y).$$ (18)

**Proof:** Modulo some minor modifications, Theorem 2 can be proven as [1, Theorem 2], which considered the generalized terminal equality constraint setting

In [1], we proposed several specific update rules $B$ fulfilling the conditions of Theorems 1 and 2. The interested reader is referred to [1] for a detailed description and discussion of these update rules.

**III. PERFORMANCE ANALYSIS OF ECONOMIC MPC WITH SELF-TUNING TERMINAL COST**

Loosely speaking, Theorem 2 means that $\lambda_\infty$ is equal to the best robustly achievable steady-state cost of the $\omega$-limit set of the resulting closed-loop system. While this is a desirable behavior of the closed-loop system, these results are of rather conceptual nature, in the sense that no a priori verifiable bounds for $\lambda_\infty$ are given. As a main result of this paper, we now show that when using the generalized terminal region constraint as in (4)–(5), $\lambda_\infty$ is a local minimum of the stage cost on the feasible steady-state set, i.e., to the following optimization problem:

\[
\min_{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m} \ell(x,u). \quad (19)
\]

Let $\Omega_\infty := \{(\bar{x}, \bar{u}) \in \mathbb{R}^n : \exists t_n \to +\infty \text{ s.t. } x(t_n) = \bar{x}, \|x(t_n) - \bar{x}\| \to 0\}$ and the fact that in both cases, at least one steady-state $(\bar{x}, \bar{u}) \in \Omega_\infty$ is a local minimizer of problem (19).

We obtain the following corollary from Theorem 3.

**Corollary 1:** Suppose that the conditions of Theorem 3(i) or (ii) are satisfied. Then $\lambda_\infty$ is a local minimum of problem (19).

**Proof:** The result immediately follows from Theorem 3 by noting that $\ell(\bar{x}, \bar{u}) = \lambda_\infty$ for all $(\bar{x}, \bar{u}) \in \Omega_\infty$.

In order to prove Theorem 3, we need the following auxiliary result.

**Lemma 1:** Let $\epsilon := (1 - \lambda_{\min}(Q)/\lambda_{\max}(P))/\lambda_{\min}(P)$. For each steady-state $(\bar{x}, \bar{u}) \in \mathbb{R}^n$ and each $x \in \mathbb{R}^n$, it holds that $x^+ = f(x, \kappa_f(x, \bar{x})) \in \mathbb{R}^n(y)$, for all $y \in B_\epsilon(\bar{x})$ with

\[
\epsilon := \left(-\sqrt{\epsilon + \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)^2}}\right)^2. \quad (20)
\]

The proof of Lemma 1 is omitted in this conference paper due to space limitations. It is based on the fact that according to Assumption 1(ii), the terminal region is contractive under application of the auxiliary terminal control law.

**Proof of Theorem 3:** (i) Assume for contradiction that there exists $(\bar{x}, \bar{u}) \in \Omega_\infty$ which is not a local minimizer of problem (19). This means that for each $\delta > 0$, there exists another steady-state $(x', u') \in B_\delta(\bar{x}, \bar{u}) \cap \mathbb{Z}$ with $\ell(x', u') < \ell(\bar{x}, \bar{u})$. Fix some $\delta \leq \epsilon/2$, with $\epsilon$ as in (20). By definition of $\Omega_\infty$, there exists an infinite sequence of time instants $\{t_i\}$ such that $(\bar{x}^*(t_i), \bar{u}^*(t_i)) \in B_\delta(\bar{x}, \bar{u}) \cap \mathbb{Z}$. Then, at the respective next time instant $t_i + 1$, a feasible solution to problem (4)–(5) is given by the end piece of the previously optimal solution appended by the terminal controller, i.e.,
\( \tilde{u}(t+1) := [u^*(1|t)]^T, \ldots, \kappa_f(x^*(N|t), \tilde{x}^*(t))]^T \), and
\( \tilde{x}(t+1), \tilde{u}(t+1) := (\tilde{x}^*(t), \tilde{u}^*(t)) \). The corresponding feasible predicted state sequence is \( x(t+1) := [x^*(1|t)]^T, \ldots, f(x^*(N|t), \kappa_f(x^*(N|t), \tilde{x}^*(t))]^T \).

By Lemma 1, it follows that
\[ f(x^*(N|t), \kappa_f(x^*(N|t), \tilde{x}^*(t))) \in \mathcal{X}(y), \]
for all \( y \in B_x(x^*(t)) \) with \( \epsilon \) given by (20). As \( \delta \leq \varepsilon/2 \) and hence \( x' \in B_x(x^*(t)) \), it follows that
\[ f(x^*(N|t), \kappa_f(x^*(N|t), \tilde{x}^*(t))) \in \mathcal{X}'(y). \]
But then another feasible solution to problem (4)-(5) at each time \( t+1 \) is given by \( \tilde{u}(t+1) \) and \( \tilde{x}(t+1), \tilde{u}(t+1) := (x^*, u^*) \). Thus, by definition of \( \ell_{\min} \) in (9), we obtain
\[ \lim_{t \to \infty} \ell_{\min}(x(t)) \leq \ell(x, u^*) < \ell(\tilde{x}, \tilde{u}) = \lambda_\infty. \]
This is a contradiction to the fact that \( \lambda_\infty = \lim_{t \to \infty} \ell_{\min}(x(t)) \) as shown in Theorem 2(ii).

(ii) Assume for contradiction that each \( (\tilde{x}, \tilde{u}) \in \Omega_\infty \) is not a local minimizer of problem (19). This means that for each \( \delta > 0 \) and each \( (\tilde{x}, \tilde{u}) \in \Omega_\infty \), there exists another steady-state \( (x', u') \in B_\delta(\tilde{x}, \tilde{u}) \cap \mathbb{Z} \) such that \( \ell(x', u') < \ell(\tilde{x}, \tilde{u}) \).

Suppose that the system map \( f(x, u) = Ax + Bu \) with \( (A, B) \) stabilizable, and choose any controller such that \( A_K := A + BK \) is stable. Next, for a given \( Q > 0 \), one can compute the matrix \( P \) as the solution to the Lyapunov equation \( A_K^TPA_K - P = -Q \). Then, with \( \kappa_f(x, \tilde{x}) := K(x - \tilde{x}) + \tilde{u} \), requirement (ii) of Assumption 1 is satisfied for each \( \tilde{x} \). Now define \( \ell(x) := \ell(x, K(x - \tilde{x}) + \tilde{u}) - \ell(\tilde{x}, \tilde{u}) \). By a straightforward modification of [3, Lemma 22], it follows that there exists some \( H \) such that \( H - \ell_{xx}(x) \geq 0 \) for all \( x \in \mathbb{Z}_K \) and \( (\tilde{x}, \tilde{u}) \in \Omega \). Following the same procedure as in [3, Section 4.1], one can show that requirement (iii) of Assumption 1 is satisfied for all \( (\tilde{x}, \tilde{u}) \in \mathbb{Z} \) and all \( x \in \mathcal{X}(\tilde{x}) \) if the terminal cost \( V_f \) is defined as
\[
V_f(x, \tilde{x}) = 1/2(x - \tilde{x})^T G(x - \tilde{x}) + \ell_{x}(\tilde{x})^T(I - A_K)^{-1}(x - \tilde{x}),
\]
where \( G \) is the solution to the Lyapunov equation \( A_K^TG A_K - G = -H \).

Finally, it remains to determine the set \( \mathcal{Z} \) such that condition (i) in Assumption 1 can be satisfied for all steady-states \( (\tilde{x}, \tilde{u}) \in \mathbb{Z} \), i.e., such that state and input constraints are satisfied in the terminal region around each steady-state \( (\tilde{x}, \tilde{u}) \in \Omega \). Let \( \mathcal{X}(\tilde{x}) := \{x \in \mathcal{X}(\tilde{x}) | x \neq 0\} \). Due to the definition of the terminal region \( \mathcal{X}(\tilde{x}) \) and the terminal controller \( \kappa_f \), we have that \( \mathcal{X}(\tilde{x}) = \{\tilde{x}\} \oplus \mathcal{X}(\tilde{x}) \). Hence condition (i) in Assumption 1 is satisfied if the set \( \mathcal{Z} \) is defined as
\[
\mathcal{Z} = \mathcal{Z} \cap \mathcal{X}(\tilde{x}) \cap K \mathcal{X}(\tilde{x}).
\]

IV. CALCULATION OF TERMINAL INGREDIENTS

We now show how the terminal ingredients can be calculated such that Assumption 1 is satisfied. The construction is built on the results in [3], where a terminal region around a fixed steady-state is constructed in case that both \( f \) and \( \ell \) are twice continuously differentiable, which we assume in the following. In this conference paper, we only consider the case of linear systems and refer to a forthcoming journal publication for possible extensions to nonlinear systems [16].

Remark 4: Some comments on the terminal regions \( \mathcal{X}(\tilde{x}) \) and the set \( \mathcal{Z} \) are in order. Recalling the definition of \( \mathcal{X}(\tilde{x}) := \{x \in \mathbb{R}^n : E(x, \tilde{x}) \leq \alpha \} \) from Assumption 1, one can see that for \( \alpha \to 0 \), it holds that \( \mathcal{Z} \) approaches \( \mathcal{Z} \). Hence in order to obtain the strongest results in Theorem 3, it would be desirable to choose \( \alpha \) small. Also, one can see that for \( \alpha \to 0 \), the feasible set of problem (4)-(5) approaches the feasible set of problem (2)-(3). On the other hand, larger values of \( \alpha \), i.e., larger terminal sets, might result in a larger feasible set of problem (4)-(5). In fact, instead of using a constant \( \alpha \) for the definition of the terminal regions \( \mathcal{X}(\tilde{x}) \), one can also define \( \alpha \) as a function of \( \tilde{x} \) given by
\[
\alpha(\tilde{x}) = \max_{(x, \kappa_f(x, \tilde{x})) \in \mathcal{Z}} \alpha, \quad \alpha \in \mathbb{Z} \cap \mathcal{X}(\tilde{x}), \quad \alpha \in \mathbb{Z} \cap \mathcal{X}(\tilde{x}).
\]
in which case \( \bigcup_{\tilde{x} \in \mathbb{Z}_K} \mathcal{X}(\tilde{x}) \) can be seen as an ellipsoidal approximation of the maximal invariant set for tracking (projected on \( \mathcal{X} \)) as used in [11, 12]. Then, condition (i) in Assumption 1 is satisfied for \( \mathcal{Z} = \mathcal{Z} \) by definition of \( \alpha(x) \). Furthermore, as \( \alpha(x) \) is continuous in \( \tilde{x} \) as shown in (23) as \( \kappa_f \) is continuous, for any compact set \( \mathcal{Z} \subset \text{int}(\mathbb{Z}) \) one can modify Lemma 1 appropriately and hence Theorem 3 is valid with \( \mathcal{Z} \) in (19) being any compact set \( \mathcal{Z} \subset \text{int}(\mathbb{Z}) \). However, we note that calculating \( \alpha(x) \) might be in general be difficult (depending on the shape of \( \mathcal{Z} \) and \( \kappa_f \)). In this case, another relaxation compared to a fixed \( \alpha \) can be used. Namely, given some initial value \( \alpha_0 > 0 \) and some \( 0 < \alpha_{\min} \leq \alpha_0 \), one

4In fact, in [2] it was shown that the asymptotic average performance is at least as good as the global minimum of problem (19) with \( \mathcal{Z} = \mathbb{Z} \). In Section IV (see Remark 4) we comment on how the generalized terminal region can be chosen such that \( \mathcal{Z} \) can be any compact set \( \mathcal{Z} \subseteq \text{int}(\mathbb{Z}) \).
can, at subsequent time instants, define \( \alpha(t) = \max\{\alpha_0(1 - (1 - \theta) \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} t), \alpha_{\min}\} \) for some \( 0 < \theta < 1 \) and in (5d) use the time-varying terminal regions \( X^f(x, t) := \{x \in \mathbb{R}^n : E(x, \bar{x}) \leq \alpha(t)\} \). Then, condition (i) in Assumption 1 is satisfied for \( Z = \bar{Z}(t) := Z \cap \bigcup_{t=0}^{\infty} X^f(t) \).

Furthermore, analogous to Lemma 1, one can show that for each \( t \in I_{\geq 0}, x^+ \in X^f(y, t + 1) \) for all \( x \in X^f(x, t) \) and all \( y \in B_x(\bar{x}) \) with \( \varepsilon := \left( -\sqrt{c + \sqrt{c + \theta \lambda_{\min}(Q) \ell}}, \sqrt{\alpha(t)} \right) \). Thus, as \( \alpha_{\min} \) can be chosen arbitrarily small and for each \( \alpha_{\min} > 0 \) there exists some \( \bar{t} \) such that \( \alpha(t) = \alpha_{\min} \) for all \( t \in I_{\geq \bar{t}} \), one can again apply Theorem 3 with \( \bar{Z} \subseteq \text{int}(Z) \).

\[ \square \]

\section{V. Example}

We illustrate the obtained result with the following simple example. Consider the system (1) with \( f(x, u) = 1.3x + u \) and \( Z = [-4, 4] \times [-2, 2] \). The stage cost function is given by \( \ell(x, u) = (1/10)(x+3)(x+2)(x-2)(0.9x-3)+u^2 \). The terminal cost \( V_f \), the auxiliary terminal controllers \( \kappa_f(x, \bar{x}) \), the terminal regions \( X^f(x) = \{x : (x-\bar{x})^2 \leq 0.25\} \) and the set \( Z \) (according to (22)) were calculated such that Assumption 1 is satisfied. The set of all steady-states in \( \bar{Z} \) is \( S := \{x \in [0, 1.2] : \bar{x} = -0.3x, \bar{x} \in [-3.5, 3.5]\} \). Figure 1(a) shows the cost function \( \ell \) restricted to the steady-state set, i.e., for \( u = -0.3x \). One can see that the unconstrained (19) has a local minimum for \( x \approx -2.45 \) with \( \ell(x, -0.3x) \approx -0.033 \) and a global minimum for \( x \approx 2.65 \) with \( \ell(x, -0.3x) \approx -0.42 \).

Figures 1(b)–1(d) show closed-loop sequences of \( \lambda \) and \( x \) and \( u \) for four different initial conditions and \( N = 3 \). As guaranteed by Theorem 3, one can see that \( \lambda \) converges to one of the two minima of problem (19), depending on the initial condition. On the other hand, for long enough prediction horizons \( (N \geq 5) \), the terminal region around the optimal steady-state \( x \approx 2.65 \) is reachable from all feasible initial conditions \( x_0 \in [-4, 4] \), and hence \( \lambda_{\infty} = -0.42 \) for all \( x_0 \in [-4, 4] \) as guaranteed by Theorem 2.

\section{VI. Conclusions}

In this paper, we have shown that when using a generalized terminal region constraint instead of a generalized terminal equality constraint, improved and a priori verifiable bounds for the closed-loop average performance of an economic MPC algorithm with self-tuning terminal cost can be obtained. Namely, this average performance is at least as good as a value corresponding to a local minimum of the stage cost function restricted to the set of feasible steady-states (respectively, a global minimum in case of linear systems and convex cost and constraints). For the case of linear systems, we also discussed a procedure how the terminal ingredients for the generalized terminal region setting can be computed.

\section{REFERENCES}